Extremities

C.J. Argue

Warm Up

(ACOPS) There are n points in the plane such that the triangle formed by any three of them has area at most 1. Show that all the points are contained in a triangle with area at most 4.

Problems

1. (ACOPS) There are 50 people sitting around a circular table, and each person's age is the average of the ages of the two people next to him. Prove that everyone is the same age.

2. (ACOPS) There are finitely many circular coins on a table, all of different sizes. Show that there is a coin touching at most 5 other coins.

3. Find all positive solutions to $x_1 + x_3 = x_2^2$, $x_2 + x_4 = x_3^2$, $x_3 + x_5 = x_4^2$, $x_4 + x_1 = x_5^2$, $x_5 + x_2 = x_1^2$.

4. (ACOPS) Prove that for any arrangement of the numbers $1, 2, \ldots, n^2$ on the cells of a $n \times n$ chessboard, there are two (horizontally, vertically, diagonally) adjacent squares whose values differ by at least $n + 1$.

5. n identical cars are stopped on a circular track. Together, they have enough gas to complete one lap. Show that one of the cars can complete a lap by collecting fuel from the other cars on the way.

6. There are n points in the plane that are not all collinear. Show that there is a line intersecting exactly two points.

Homework

1. Suppose that S is a set of points in the plane, and that for each point $X \in S$, there are distinct points $y, z \in S$ such that x is the midpoint of the line segment yz Show that S contains infinitely many points.

2. There are n people at a party. Show that there is a way to split the people into two rooms such that each person is in the same room as at most half of his friends. Does it matter whether friendships are mutual?

3. (Putnam 79) Let S be a set of $2n$ points in the plane, n of which are colored red and the remaining n of which are colored blue. Prove or disprove: there are n line segments, no two with a point in common, such that the endpoints of each segment are points of A having different colors.