Integer Linear Equations

Number Theory

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September 25, 2016

1 Introduction

Definition 1 (Inverses in Modular Arithmetic)**.** The modular inverse of an integer $a \pmod{n}$ is denoted a^{-1}, where a^{-1} is the integer such that $aa^{-1} \equiv 1 \pmod{n}$ and $a^{-1}a \equiv 1 \pmod{n}$.

Property 2. Only numbers coprime to n (numbers that have a gcd of 1 with n) have a modular inverse $\pmod{n}$.

Example 3.

$$5 \times 2 \equiv 1 \pmod{3} \quad \Big| \quad 17 \times 3 \equiv 1 \pmod{10} \quad \Big| \quad 4 \times 7 \equiv 1 \pmod{9}$$

Definition 4 (Extended Euclidean Algorithm)**.** This extension to the Euclidean Algorithm, when given integers a, b, computes the integers x and y such that $ax + by = gcd(a, b)$.

Example 5. Apply the Extended Euclidean Algorithm to $\gcd(126, 224)$.

Solution.

$$\begin{aligned}
\gcd(126,224) &= \gcd(126, 224-126) & 224 &= 1\times 126+98 & 98 &= 224 - 1\times 126 \\
\gcd(126,98) &= \gcd(98, 126-98) & 126 &= 1\times 98+28 & 28 &= 126-(224-126) = 2\times 126 - 224 \\
\gcd(98,28) &= \gcd(28, 98-3\cdot 28) & 98 &= 3\times 28+14 & 14 &= (224-126) - 3(2\times 126-224) \\
\gcd(28,14) &= \gcd(14, 28-2\cdot 14) & 28 &= 2\times 14+0 & 14 &= 4\times 224 - 7\times 126 \\
&= \gcd(14,0)
\end{aligned}$$

Thus, $\gcd(126, 224) = \boxed{14}$ and $14 = 4 \times 224 - 7 \times 126$.

Definition 6 ((Linear) Diophantine Equation)**.** A Diophantine equation is a polynomial equation where only integer solutions are studied. A Linear Diophantine equation is one between sums of monomials of degree 0 or 1.

Example 7. $a + 2b = 1$ is a linear Diophantine equation with many solutions, like (1,0).

Example 8. $x^2 + y^2 = z^2$ has infinitely many solutions: the Pythagorean triples, such as (3,4,5).

Definition 9 (Chinese Remainder Theorem)**.** Let $n_1, n_2, \ldots, n_l$ be integers greater than 1, and denote by N the product of all the n_i. If the n_i are pairwise coprime, and if $a_1, a_2, \ldots, a_k$ are any integers, then there exists an integer x such that

$$\begin{aligned}
x &\equiv a_1 \pmod{n_1} \\
&\vdots \\
x &\equiv a_k \pmod{n_k}
\end{aligned}$$

and any two such x are congruent modulo N.

2 Problems

1. Find the inverses of 3, 4, and 5 modulo 22.

2. Characterize the solutions to $7x + 3y = 8$

3. We have a 7 gallon jug and an 11 gallon jug. How can we measure out 1 gallon of water?

4. Misha only feeds squirrels in groups of 4 or 7. What is the largest number of squirrels that Misha cannot feed at once? Given that the sizes of the groups of squirrels are relatively prime, can you generalize?

5. Two farmers agree that pigs are worth 300 dollars and that goats are worth 210 dollars. When one farmer owes the other money, he pays the debt in pigs or goats, with "change" received in the form of goats or pigs as necessary. (For example, a 390 dollar debt could be paid with two pigs, with one goat received in change.) What is the amount of the smallest positive debt that can be resolved in this way? [1]

6. Victor has stolen Misha's pizzas! He will only give them back if Misha can tell him exactly how many he stole. However, Misha only remembers that when piled in stacks of 2, there was 1 left over, when piled in stacks of 5, there were 3 left over, and when in stacks of 7, there were 2 left over. He also remembers there were not *TOO* many pizzas. How many pizzas has Victor stolen?

7. Find all solutions of $x^3 - x + 1 \equiv 0 \pmod{35}$.

8. Find all non-negative integer solutions $(n_1, \ldots, n_{14})$ to [2]
$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1599$$

9. Ninety-four bricks, each measuring $4'' \times 10'' \times 19''$, are to stacked one on top of another to form a tower 94 bricks tall. Each brick can be oriented so it contributes $4''$ or $10''$ or $19''$ to the total height of the tower. How many different tower heights can be achieved using all ninety-four of the bricks? [3]

10. How many x with $38 \leq x \leq 289$ satisfy $3x \equiv 8 \pmod{11}$? [4]

11. Find all integer solutions to $3456x + 346y = 234$. [4]

3 Challenge Problems

1. Prove that $x^4 + y^4 = z^2$ has no integer solutions.

2. On the real number line, paint red all points that correspond to integers of the form $81x+100y$, where x and y are positive integers. Paint the remaining integer point blue. Find a point P on the line such that, for every integer point T, the reflection of T with respect to P is an integer point of a different colour than T. [5]

[1] AMC 2006
[2] From USAMO 1979
[3] AIME 1994
[4] From *Number Theory for Mathematical Contests* by David A. Santos
[5] India TST

3. What is the largest positive integer that cannot be expressed as a sum of non-negative integer multiples of 13, 17, and 23? [6]

4. Determine the maximum value of $m^2 + n^2$, where m and n are integers satisfying $m, n \in \{1, 2, \ldots, 1981\}$ and $(n^2 - mn - m^2)^2 = 1$. [7]

5. Observe that

$$2^2 + 3^2 + 6^2 = 7^2$$
$$3^2 + 4^2 + 12^2 = 13^2$$
$$4^2 + 5^2 + 20^2 = 21^2$$

   (a) Find integers x and y so that $5^2 + 6^2 + x^2 = y^2$.

   (b) Conjecture a general rule that is being illustrated here.

   (c) Prove your conjecture. [8]

6. Prove the conjecture you found for Problem 4. (This is called the Chicken McNugget Theorem or the Postage Stamp Problem or the Frobenius Coin Problem among many other names.)

[6] PUMaC 2013
[7] IMO 1981
[8] UNCO 1993