Graph Theory (Coloring)

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1 Introduction

Definition 1 (Graph)**.** A (simple) graph consists of vertices/nodes and (undirected) edges connecting pairs of distinct vertices, where there is at most one edge between a pair of vertices.

Definition 2 (Degree)**.** The degree of a vertex is the number of edges through a vertex.

Definition 3 (Neighbor)**.** A vertex u is a neighbor of a vertex v if there is an edge between u and v.

Definition 4 (Paths and Cycles)**.** A path is a sequence of vertices where consecutive vertices are connected by an edge. A cycle is a path starting and ending at the same vertex.

Definition 5 (Proper (Vertex) Coloring)**.** In a proper vertex coloring of a graph, every vertex is assigned a color and if two vertices are connected by an edge, they must have different colors. If a graph can be colored with k colors, it is called k-colorable.

Definition 6 (Chromatic Number)**.** The chromatic number of a graph G, denoted $\chi(G)$ is the least number of colors required to properly color the vertices of a graph.

Proposition 7. A graph is 2-colorable if and only if it does not contain an odd cycle

1.1 Graphs of Large Chromatic Number

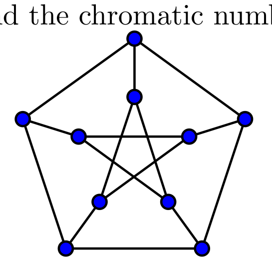
The Zykov Graphs are a recursively defined family of graphs. Z_1 has one point. Then Z_{i+1} is defined by taking i copies of Z_i, adding a set A_i of $|V(Z_i)|^i$ (the number of vertices of Z_i raised to i) vertices where we connect each vertex in A_i to one vertex in each copy of Z_i in a different way.

Proposition 8. The Zykov graphs do not contain a triangle (a cycle of length 3) and have the property that $\chi(Z_i) = i$.

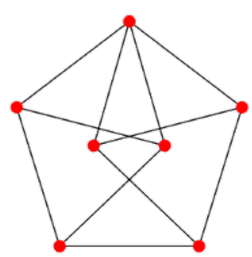
Remark 9. The Zykov graphs are an example which show that triangles aren't the only thing which cause chromatic numbers to be large!

2 Problems

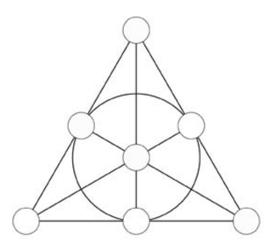
1. Find the chromatic number of:



(a)



(b)



(c)

2. Jim has six children, and it is not an easy bunch. Chris fights with Bob, Faye, and Eve all the time; Eve also fights with Al and Di; and Al and Bob fight all the time. Can Jim put the children in two rooms so that pairs of fighters are in different rooms? If so, show how.[1]

[1] *Discrete Mathematics* by Lovász, Pelikán, Vesztergombi

3. Does there exist a graph with the following degrees (if so, draw it): (a) $2, 2, 3, 3, 4, 4$; (b) $0, 2, 2, 2, 4, 4, 6$; (c) $2, 2, 3, 3, 4, 4, 5$?[1]

4. A complete graph on n vertices (denoted K_n) has the property that there is an edge between any two vertices. Show that $\chi(K_n) = n$.

5. Find a general way to construct a 3-chromatic graph such that the size of the smallest cycle is n, where n is some natural number.

6. If a graph has n vertices and is 2-colorable, find the maximum number of edges it can have.

7. Prove the Handshaking Lemma: the sum of the degrees of the vertices of a graph is twice the number of edges.

8. Prove that there must be two vertices of equal degree in any graph.

9. Prove that if every vertex has degree at least 2 then the graph has a cycle.

10. If all vertices of a graph have degree at most d and there exists a vertex with degree less than d, prove that G is d-colorable.[1]

11. Assuming friendship is mutual between pairs of people, given 6 people, show that there are either 3 people who are all friends or 3 people who are all not friends.

3 Challenge Problems

1. The Mycielski graphs are a family of graphs similar to the Zykov graphs. Again, M_1 is a one point, and M_2 is two vertices connected by an edge. Then, to define M_{i+1}, we have a copy of M_i and then we add a "twin vertex" for every vertex in M_i, and we connect this twin vertex to all the neighbors of the corresponding vertex in M_i. Finally, we add one more vertex which we connect to every twin vertex.

   Prove that the Mycielski graphs have no triangles and that $\chi(M_i) = i$.

3.1 Hypergraphs!!

Definition 10 (Hypergraph)**.** A hypergraph consists of a set of vertices and edges which can connect any number of vertices (instead of just 2)

Definition 11 (Linear Space)**.** A linear space is a hypergraph where any pair of distinct vertices is contained in precisely one edge. It is called trivial if one edge contains all vertices and a near pencil if one edge contains all but one vertex.

Theorem 12 (de Bruijn-Erdös)**.** If a non-trivial linear space has n vertices and m edges, then $m \geq n$.

2. Show that there is no linear space with 2016 vertices such that every edge contains either 11 or 12 vertices.

3. Prove that if the edges of a complete graph K_n are colored with $m \geq 2$ colors such that each color forms a complete subgraph, then $n \geq m$.