ARML 1995 Individual Round

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October 2, 2016

Problems 1-2

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1. Compute the largest prime factor of

$$(3\cdot(3\cdot(3\cdot(3\cdot(3\cdot(3\cdot(3\cdot(3\cdot(3\cdot(3\cdot(3+1)+1)+1)+1)+1)+1)+1)+1)+1)+1)+1)$$

2. $\triangle ABC$ is similar to $\triangle MNP$. If $BC = 60, AB = 12, MP = 8$, and $BC > AC > AB$, compute the sum of all possible integer values for the ratio of the area of $\triangle ABC$ to the area of $\triangle MNP$.

Solutions 1-2

1. A reasonable solution is to simply do the computation which is doable, and gives $265720 = 2^3 * 5 * 7 * 13 * 73$. There are also several ways to notice that this expression equals 111111111111_3 (there are 12 1's).

$$111111111111_3 = 11_3 * 10101_3 * 1000001_3 = 4 * 91 * 730 = 2^3 * 5 * 7 * 13 * 73$$

This solution can be slightly faster, and the answer is $\boxed{73}$.

2. (Diagram on board) Let x denote the length of AC. Then, since triangles are 2-dimensional, the ratio of the ratio of the areas of $\triangle ABC$ to $\triangle MNP$ is $(x/8)^2$. We're given $x < 60$ and $48 < x$ by the triangle inequality, so $36 < (x/8)^2 < 56.25$, and the sum of the possible ratios is $37 + \cdots + 56 = 93 \cdot 20/2 = \boxed{930}$

Problems 3-4

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3. Determine all positive primes p such that $p^{1994} + p^{1995}$ is a perfect square.

4. Let $.\overline{A} = .AAA\ldots$ Compute the number of distinct ordered triples (A, B, C) with $A, B, C \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that k is an integer if

$$k = \frac{.\overline{ABC} + .\overline{ACB} + .\overline{BAC} + .\overline{BCA} + .\overline{CAB} + .\overline{CBA}}{.\overline{A} + .\overline{B} + .\overline{C}}$$

Solutions 3-4

3. We factor $p^{1994} + p^{1995} = p^{1994}(p+1)$. $p^{1994} = (p^{997})^2$ is always a perfect square, so we just need $p+1$ to also be a perfect square. Suppose $p+1$ is a perfect square, and let $s^2 = p+1$. Then, $s^2 - 1 = p$, and $(s-1)(s+1) = p$. p is prime, so $s-1 = 1$ or $s+1 = 1$. Therefore $2^2 - 1 = \boxed{3}$ is the only solution.

4. We know $.\overline{ABC} = ABC/999$ and $.\overline{A} = A/9$. We express everything as a fraction.

$$k = \frac{ABC + ACB + BAC + BCA + CAB + CBA}{111(A+B+C)} = \frac{222(A+B+C)}{111(A+B+C)}$$

This is an integer except when $A = B = C = 0$, so the answer is $10^3 - 1 = \boxed{999}$.

Problems 5-6

Problems 5-6

5. Compute the number of distinct planes passing through at least three vertices of a given cube.

6. Determine all positive integers $k \leq 2000$ for which $x^4 + k$ can be factored into two distinct trinomial factors with integer coefficients.

5. We need to be careful. Let us break up the planes based on how many vertices of the bottom face they pass through. 1 plane passes through exactly 4 vertices, and 0 pass through exactly 3 vertices. The planes passing through 2 vertices of the bottom face either pass through an edge or a diagonal of the bottom face. Each edge on the bottom face has 2 valid planes passing through it and each diagonal has 3 valid planes, for a total of $4 \cdot 2 + 2 \cdot 3 = 14$. The valid planes passing through exactly 1 vertex of the bottom face must intersect the top face on a specific diagonal, so there are 4 such planes. There is 1 plane passing through 0 vertices, so this makes a total of $1 + 14 + 4 + 1 = \boxed{20}$ planes.

6. Let the two trinomials be $x^2 + ax + b$ and $x^2 + cx + d$. This gives us the equations $a + c = 0, ac + b + d = 0, ad + bc = 0, bd = k$. Substitute $c = -a$ in the second and third equations to be $b + d = a^2, a(d - b) = 0$. If $a = 0$ then $bd \leq 0$ giving us no valid k. Otherwise $b = d$, and a^2 is even, so $k \leq 2000$ gives $a^2 = 4, 16, 36, 64 \to k = \boxed{4, 64, 324, 1024}$.

Problems 7-8

Problems 7-8

7. Let $f(x) = \sqrt{x+2} + c$. Determine all real values of c such that the graphs of $f(x)$ and its inverse $f^{-1}(x)$ intersect at two distinct points.

8. For x and y in radians, compute the number of solutions in ordered pairs (x, y) to the following system:

$$\begin{cases} \sin(x+y) = \cos(x-y) \\ x^2 + y^2 = \left(\frac{1995\pi}{4}\right)^2 \end{cases}$$

Solutions 7-8

7. It is helpful to draw some graphs. Some should be on the blackboard. If $f(x)$ and $f^{-1}(x)$ intersect at some (a, b), then $f(a) = a = b$. Let's find the points where this happens. We have the equation $x = \sqrt{x+2} + c \to x^2 + (-2c-1)x + c^2 - 2 = 0$. The discriminant of this is equal to $4c + 9$, so for there to be two solutions we need $c > -9/4$. The graphs show us that we also need $c \le -2$, so the answer is $\boxed{(-9/4, -2]}$.

8. Make the substitutions $a = x + y$ and $b = x - y$. Then, $\sin(a) = \cos(b)$, so $b = \pm(a - \pi/2) + 2k\pi$ for some $k \in \mathbb{Z}$. We substitute this in for b in the second equation which is now $a^2 + b^2 = 2\left(\frac{1995\pi}{4}\right)^2$ to get

$$2a^2 + (\pm 4k\pi - \pi)a + \frac{\pi^2}{4} + 4k^2\pi^2 \mp 2k\pi^2 - \frac{1995^2\pi^2}{8} = 0.$$

The discriminant of this is $\pi^2((1995 - 4k)(1995 + 4k) \pm 8k - 1)$. This is greater than 0 when $-498 \le k \le 498$ and equal to 0 for $+, k = 499$ and $-, k = -499$, so the total number of solutions is $2 * (2 * 997 + 1) = \boxed{3990}$.