Practice Math Contest — Misha Lavrov

The Largest Prime Factor Function: Solutions

Western PA ARML Practice — *January 22, 2017*

In the problems[1] below, let $P(n)$ denote the largest prime factor of n. For example, since $2016 = 2^5 \cdot 3^2 \cdot 7$, $P(2016) = 7$; since 2017 is prime, $P(2017) = 2017$.

1. (a) Find $P(100! + 101!)$.

   Answer: 97. We have $100! + 101! = 100! \cdot (1 + 101) = 1 \cdot 2 \cdot 3 \cdots 99 \cdot 100 \cdot 102$. Of these factors, 97 is the largest which is prime, and all of the composite factors are less than $97 \cdot 2$, so they can't themselves have a prime factor greater than 97.

   (b) Find the largest *2-digit* prime factor of $\binom{200}{100}$.

   Answer: 61. We can write $\binom{200}{100}$ as $\frac{200!}{100!^2}$. For every prime number p between 67 and 99, it will divide 200! twice (once for the factor of p and once for the factor of $2p$), and 100! once, so it will not divide $\frac{200!}{100!^2}$. The largest prime number smaller than 67 is 61, which divides 200! three times (from the factors of 61, 122, and 183) and 100! only once, so it divides $\frac{200!}{100!^2}$ once.

2. Prove that there are infinitely many integers n such that $P(n) < P(n+1) < P(n+2)$.

   Proof: Choose an arbitrary odd prime p. Since $p^2 - 1 = 2^2 \cdot \frac{p-1}{2} \cdot \frac{p+1}{2}$, we have $P(p^2-1) \le \frac{p+1}{2} < p$, so $P(p^2-1) < P(p^2)$.

   If $P(p^2+1) > p$, then $P(p^2-1) < P(p^2) < P(p^2+1)$, and we have found one such triple. Otherwise, let $k > 0$ be the smallest integer such that $p^{2^k} + 1$ has a prime factor larger than p. By assumption, $P(p^{2^k}+1) > P(p^{2^k})$, and we can factor $p^{2^k} - 1$ as $(p^2-1)(p^2+1)(p^4+1)\cdots(p^{2^{k-1}}+1)$, each of which has no prime factors larger than p, again by assumption. Therefore $P(p^{2^k}-1) < P(p^{2^k}) < P(p^{2^k}+1)$, and we have found one such triple.

   We get a different triple for every odd prime p we choose, since the middle number will always be a power of p, so we can find infinitely many such triples.

3. Prove that there are infinitely many triples of distinct positive integers (a, b, c) such that $P(a^2+1) = P(b^2+1) = P(c^2+1)$.

   Proof: Let k be an arbitrary positive integer. Define $p = P((2k-1)^2+1)$, $q = P((2k)^2+1)$, and $r = P((2k+1)^2+1)$.

   Note that, for any x, since $(x+i)(x+1-i) = x(x+1)+1+i$ and $(x-i)(x+1+i) = x(x+1)+1-i$, we have $(x^2+1)((x+1)^2+1) = (x(x+1)+1)^2+1$.

   Similarly, since $(x+i)(x+2-i) = x(x+2)+1+2i$ and $(x-i)(x+2-i) = x(x+2)+1-2i$, we have $(x^2+1)((x+2)^2+1) = (x(x+2)+1)^2+4$. When x is odd, this is equal to $4((\frac{x(x+2)+1}{2})^2+1)$.

[1] Problems 1(a) and 1(b) are taken from posts on the Art of Problem Solving forum, with slight modification. Problems 2 and 3 are taken from posts on `http://www.reddit.com/r/mathriddles/`.

This was all just motivation for the following calculations:

$$\begin{aligned}
((2k-1)^2+1)((2k)^2+1) &= ((2k-1)(2k)+1)^2+1 \\
&=: x^2+1 \\
((2k)^2+1)((2k+1)^2+1) &= ((2k)(2k+1)+1)^2+1 \\
&=: y^2+1 \\
((2k-1)^2+1)((2k+1)^2+1) &= 4((2k^2)^2+1) \\
&=: 4(z^2+1).
\end{aligned}$$

Therefore $P(x^2+1) = \max\{p,q\}$, $P(y^2+1) = \max\{q,r\}$, and $P(z^2+1) = \max\{p,r\}$.

Among $P((2k-1)^2+1), P((2k)^2+1), P((2k+1)^2+1), P(x^2+1), P(y^2+1), P(z^2+1)$, the value $\max\{p,q,r\}$ occurs at least three times. This yields one of the solutions we wanted.

Taking arbitrarily large values of k yields triples (a,b,c) with a,b,c distinct and each at least $2k-1$ that satisfy $P(a^2+1) = P(b^2+1) = P(c^2+1)$. So arbitrarily large solutions—and therefore infinitely many solutions—exist.