Combinatorics 2

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1 Notes on Probability

1.1 Introduction

The probability of an event, denoted $P(A)$ for the probability of event A, is the "chance" that A occurs. A can be thought of as a subset of all potential outcomes that can happen Ω, so $A \subseteq \Omega$. Probabilities are always between 0 and 1, so $0 \leq P(A) \leq 1$ for all events A. If you're confused, it might help to think about what the outcomes and events are.

1.2 Relationship to Counting

If the process/problem can be broken up into outcomes all with equal probability, then the probability of some event A is the number of outcomes satisfying A divided by the total number of outcomes. Example problem: What is the probability that the sum of the rolls of 4 6-sided dice is 20? Each of the 6^4 combinations of 4 rolls are equally likely, so we can use stars and bars to find the number of these rolls which sum to 20 and then divide this by 6^4.

Similar to the principle of inclusion-exclusion, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{1}$$

We can also do "casework", and view it as a special case of the principle of inclusion-exclusion where $P(A \cap B) = 0$. Similarly, "complementary counting" gives us

$$P(A) = 1 - P(A^{\mathsf{c}}) \tag{2}$$

1.3 Independence

Events A and B are said to be *independent* if $P(A \cap B) = P(A) \cdot P(B)$. This will generally be the case in problems; however, this cannot in general be assumed. Intuitively, two events A, B are independent if knowing whether or not A occurred doesn't give any information about whether or not B occurred (and vice versa). As an example, let X and Y be independent coin tosses, Z_1 be the the number of heads, and $Z_2 = Z_1 \mod 2$. Of the pairs $(X, Z_1), (Y, Z_2), (Z_1, Z_2)$, which are independent and which are not?

1.4 Conditioning

We define the "probability of A given B", denoted $P(A|B)$, as $P(A \cap B)/P(B)$. Intuitively, this represents the probability that A occurs if we know that B happened. Also, we have $P(A) = P(A|B)P(B) + P(A|B^{\mathsf{c}})P(B^{\mathsf{c}})$. (Exercise: prove this). This can be a powerful way to do casework to figure out $P(A)$.

2 Problems

1. (Classic) You meet a man on the street and he says, "I have two children and one is a son born on a Tuesday." What is the probability that the other child is also a son?

2. Victor is summoning heroes. He notices that one of ten red heroes appear with probability $\frac{2}{3}$ and one of ten blue heroes appear with probability $\frac{1}{3}$. Given that three red heroes and eight blue heroes are desirable, what is the probability that a randomly appearing hero will be desirable?

3. Victor continues summoning. He spawns seven times, knowing that he gets either a $2\star$ or $3\star$ character with equal probability. Given that he spawns more $2\star$ characters than $3\star$ characters, what is the probability that he summons his first $3\star$ character on his third summon?

4. (AMC) A coin is altered so that the probability that it lands on heads is less than $\frac{1}{2}$ and when the coin is flipped four times, the probability of an equal number of heads and tails is $\frac{1}{6}$. What is the probability that the coin lands on heads?

5. (AMC) Bernardo randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and arranges them in descending order to form a 3-digit number. Silvia randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and also arranges them in descending order to form a 3-digit number. What is the probability that Bernardos number is larger than Silvias number?

6. (SMT) Nick has a terrible sleep schedule. He randomly picks a time between 4 AM and 6 AM to fall asleep, and wakes up at a random time between 11 AM and 1 PM of the same day. What is the probability that Nick gets between 6 and 7 hours of sleep?

7. (AMC) Suppose a and b are single-digit positive integers chosen independently and at random. What is the probability that the point (a, b) lies above the parabola $y = ax^2 - bx$?

8. (AIME 2001) A fair die is rolled four times. Determine the probability that each of the final three rolls is at least as large as the roll preceding it.

9. (HMMT 2012) Brian has a 20-sided die with faces numbered from 1 to 20, and George has three 6-sided dice with faces numbered from 1 to 6. Brian and George simultaneously roll all their dice. What is the probability that the number on Brians die is larger than the sum of the numbers on Georges dice?

10. (AMC) Two circles of radius 1 are to be constructed as follows. The center of circle A is chosen uniformly and at random from the line segment joining $(0, 0)$ and $(2, 0)$. The center of circle B is chosen uniformly and at random, and independently of the first choice, from the line segment joining $(0, 1)$ to $(2, 1)$. What is the probability that circles A and B intersect?

11. (AIME 2001) Each unit square of a 3-by-3 unit-square grid is to be colored either blue or red. For each square, either color is equally likely to be used. Find the probability of obtaining a grid that does not have a 2-by-2 red square.

12. (AMC) A parking lot has 16 spaces in a row. Twelve cars arrive, each of which requires one parking space, and their drivers chose spaces at random from among the available spaces. Auntie Em then arrives in her SUV, which requires 2 adjacent spaces. What is the probability that she is able to park?

13. (CMIMC 2016) Shen, Ling, and Ru each place four slips of paper with their name on it into a bucket. They play the following game: slips are removed one at a time, and whoever has all of their slips removed first wins. Shen cheats, however, and adds an extra slip of paper into the bucket, and will win when four of his are drawn. What is the probability that Shen wins?

14. (AIME 2007) Let S be a set with six elements. Let P be the set of all subsets of S. Subsets A and B of S, not necessarily distinct, are chosen independently and at random from P. Determine the probability that B is contained in at least one of A or $S \setminus A$.

15. (AIME 2011) The probability that a set of three distinct vertices chosen at random from among the vertices of a regular n-gon determine an obtuse triangle is $\frac{93}{125}$. Find all possible values of n.

3 More Problems

1. (ISL 2010) Compute the number of ways to place 2500 chess kings on a 100×100 board such that the following conditions hold:
   - No two kings can attack each other.
   - Each row and column contains exactly 25 kings.

2. (CGMO 2013) In a group of m boys and n girls, every pair of people either know each other or don't know each other. Suppose that, for every group of two boys and two girls, there exists a boy and a girl in that group who don't know each other. Prove that the number of pairs of a boy and a girl who know each other does not exceed $m + \frac{n(n-1)}{2}$.