2004 St. Petersburg Math Olympiad

Selected problems

(1) Roma picked a natural number n. He chose a divisor of n, multiplied it by 4, and subtracted that result from n, getting 11. What is n? Find all possible answers and prove that there are no others.

(2) A country uses coins with values of 1, 2, 3, 5, 8, 10, 15, 20, 25, 32, 50, 57, 75, and 100 cents. A machine can trade a coin for exactly four coins of the same total value (for example, a 100 cent coin for 57, 20, 20, and 3 cent coins). By using several such exchanges, is it possible to turn a 100 cent coin into 100 1-cent coins?

(3) Is it possible to fill a 5×8 grid with 1s and 3s such that in each row and column, the sum is divisible by 7?

(4) A six-digit number is called *almost lucky* if three of its digits have the same sum as the other three. (For example, 013 725 is almost lucky, since $1+3+5 = 0+7+2$.) Kostya buys two bus tickets with consecutive six-digit numbers, and finds that they are both almost lucky. Prove that one of them must end in 0.

(5) Prove that there are fewer than 500 000 almost lucky numbers (as defined in problem 4).

(6) In quadrilateral $ABCD$, let K be the midpoint of AB; let L be the midpoint of BC; let M be the point on CD such that $CM : MD = 2 : 1$. If $DK \parallel BM$ and $AL \parallel CD$, prove that $AD \parallel BC$.

(7) In a group of 35 students, each is studying the same 10 subjects, and receives a numerical final grade in each: an integer from 1 to 5. (A 1 or 2 is a failing grade; 3, 4, and 5 are passing.)

If the average grade in each subject is greater than $4\frac{2}{3}$, prove that at least 5 students did not fail any subjects.

(8) Real numbers $x, y > 0$ satisfy the condition $|4 - xy| < 2|x - y|$. Prove that one of x and y is less than 2 and the other is greater than 2.

(9) A 17×17 grid is filled with positive numbers. In each row, the numbers form an arithmetic progression. In each column, the *squares* of the numbers form an arithmetic progression. Prove that the product of the top left and bottom right numbers is equal to the product of the numbers in the other two corners.

(10) In $\triangle ABC$ points K, L, and M are chosen on sides AB, BC, and AC, respectively, such that $\angle BLK = \angle CLM = \angle BAC$. Segments BM and CK intersect at P. Prove that the quadrilateral $AKPM$ is cyclic.