Writing Proofs — Misha Lavrov

The Invariance Principle

Western PA ARML Practice — *February 28, 2016*

Warm-up

A bag contains 99 red marbles and 99 blue marbles. Taking two marbles out of the bag, you:

- put a red marble in the bag if the two marbles you drew are the same color (both red or both blue), and
- put a blue marble in the bag if the two marbles you drew are different colors.

Repeat this step (reducing the number of marbles in the bag by one each time) until only one marble is left in the bag. What is the color of that marble?

Problems

1. (Engel[1]) An 8×8 chessboard is colored in the usual way, but that's boring, so you decide to fix this. You can take any row, column, or 2×2 square, and reverse the colors inside it, switching black to white and white to black.

   Prove that it's impossible to end up with 63 white squares and 1 black square.

2. The numbers $1, 2, \dots, 100$ are written on a blackboard. You may choose any two numbers a and b and erase them, replacing them with the single number $a + b - 1$. After 99 steps, only a single number will be left. What is it?

3. Suppose you instead replace a and b by the product $ab + a + b$. What number will be left at the end?

4. At a party, some pairs of people shake hands. We call a person *odd* who has shaken hands with an odd number of other guests. Prove that there is an even number of odd people at the party.

5. A room is initially empty. Every minute, either two people enter or one person leaves. After exactly $3^{3^{3^3}}$ minutes, could the room contain exactly $3^{3^3} + 1$ people?

6. A herd of 100 cows is divided into four pens: 10 cows in the north pen, 20 cows in the east pen, 30 cows in the south pen, and 40 cows in the west pen.

   The pens are connected through a gateway we can use to let three cows out of one pen and distribute them between the others. For instance, if we let three cows out of the south pen,

[1] Arthur Engel, *Problem-Solving Strategies*.

we end up with 11 cows in the north pen, 21 cows in the east pen, 27 cows in the south pen, and 41 cows in the west pen.

Prove that we can never use this gateway to split the herd into four equal groups, with 25 cows in each of the four pens.

7. (St. Petersburg) A teacher wrote down three positive real numbers on the blackboard and told Dima to decrease one of them by 3%, decrease another by 4%, and increase the last by 5%. Dima wrote down the results in his notebook. It turned out that he wrote down the same three numbers that are on the blackboard, just in a different order. Prove that Dima must have made a mistake.

8. (Engel) There is a positive integer in each square of a rectangular table. In each move, you may double each number in a row or subtract 1 from each number of a column. Prove that you can reach a table of zeroes by a sequence of these permitted moves.

9. (a) The integers $1, 2, \ldots, n$ are written down in that order. At each step, you may swap any two integers: for example, if $n = 6$, you can begin by changing $1, 2, 3, 4, 5, 6$ to $1, 2, 5, 4, 3, 6$ by swapping 3 and 5.

   Prove that you can never return to the original order after an odd number of swaps.

   (This is one of the more difficult problems, but also the most generally useful result, so I include hints for two[2] different[3] approaches to solving it.)

   (b) The 15-puzzle is a sliding puzzle with fifteen square tiles, numbered 1 through 15, arranged in a 4×4 square. In the late 19th century, Sam Loyd offered a \$1000 prize for anyone that could get from the configuration on the left to the configuration on the right (swapping the 14 and 15 tiles) by sliding the tiles around.

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

   Prove that this is impossible, and so the prize would never have to be paid out.

10. (Putnam 2008) Start with a sequence $a_1, a_2, \ldots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by $\gcd(a_j, a_k)$ and $\operatorname{lcm}(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop, and the final sequence does not depend on the choices made.

11. Seven squares of an 8×8 grid are shaded. At each step, we shade in each unshaded square that has at least two shaded neighboring squares (horizontally or vertically). Prove that this process cannot end in the entire grid being shaded.

[2] Approach #1: Consider the number of "inversions" (pairs of integers that are out of order) at each step.
[3] Approach #2: Begin by proving the result for the special case where only adjacent integers can be swapped.