Number Theory — Misha Lavrov

Divisibility

Western PA ARML Practice — *September 20, 2015*

Warm-up

1. (ARML 1991) Compute the smallest 3-digit multiple of 7 for which the sum of its digits is also a multiple of 7.

1 The divisors of an integer

1. (AIME 1998) A divisor of 10^{99} is chosen uniformly at random. Find the probability that it's divisible by 10^{88}.
2. Find the number of ways to write 300 as a product of three positive integers $a \cdot b \cdot c$. (The product is ordered, so $1 \cdot 3 \cdot 100$ is different from $100 \cdot 1 \cdot 3$.)
3. Call n an *everyday number* if the sum of the divisors of n (including n itself) is even. For example, 6 is an everyday number, since $1+2+3+6 = 12$, but 8 is not, since $1+2+4+8 = 15$.

   How many of the divisors of 10^{100} are everyday numbers?
4. (Well-known) Suppose you're in a hallway with 100 closed lockers in a row, and 100 students walk by. The first student opens every locker. The second student closes every other locker. The third student goes to every third locker and toggles it: opens it if it's closed, and closes it if it's open. The remaining students continue this process: the n-th student goes to every n-th locker and toggles it.

   When all 100 students have walked by, which lockers are open?
5. (ARML 1984) Find all possible values of k for which $1984 \cdot k$ has exactly 21 positive divisors.
6. Let n be of the form $2^a \cdot 3^b$ for some a and b. Prove that the sum of the divisors of n (including n itself) is at most $3n$.
7. (PUMaC 2011) The sum of the divisors of n (including n itself) is 1815. If $n = 2^a \cdot 3^b$ for some a and b, find (a, b).
8. (ARML 1979) Let $\tau(n)$ denote the number of positive divisors of n. (E.g., $\tau(12) = 6$, counting $1, 2, 3, 4, 6$, and 12 itself.) For how many positive integers $n \leq 100$ is $\tau(n)$ a multiple of 3?
9. (ARML 2014) Find the smallest positive integer n such that $214 \cdot n$ and $2014 \cdot n$ have the same number of divisors.

2 Prime factorization

1. (AIME 1991) How many reduced fractions $\frac{a}{b}$ are there such that $ab = 20!$ and $0 < \frac{a}{b} < 1$?
2. Prove that $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = a \cdot b$.
3. (USAMO 1972) Prove that for all positive integers a, b, c,
$$\frac{\gcd(a, b, c)^2}{\gcd(a, b) \cdot \gcd(a, c) \cdot \gcd(b, c)} = \frac{\operatorname{lcm}(a, b, c)^2}{\operatorname{lcm}(a, b) \cdot \operatorname{lcm}(a, c) \cdot \operatorname{lcm}(b, c)}.$$
4. Find all solutions to $x^2 + 3x = y^2$, where x and y are positive integers.
5. (Putnam 2003) Show that for each positive integer n,
$$n! = \prod_{i=1}^{n} \operatorname{lcm}\left(1, 2, \ldots, \lfloor n/i \rfloor\right).$$