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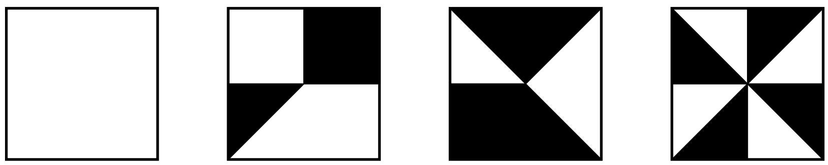
Counting Strategies

Western PA ARML Practice *December 6, 2015*

1 Warm-up

A puzzle I had when I was little contained many tiles such as the ones below: a square is divided into 8 triangles by horizontal, vertical, and diagonal lines through the center, and then some of the triangles are filled in.



No two tiles were the same, not even if you rotated them (though some were mirror images of each other). What is the largest number of tiles the puzzle could have contained?

2 Problems

1. A *para-palindrome* is a word like FOOLPROOF or BATHTUB: it's off from being a palindome by virtue of a single letter being wrong. Of course, we can also extend this to numbers, declaring 31415 to be a para-palindrome.

   How many 6-digit numbers are para-palindromes?

2. A chess piece starts out in the bottom left corner of an 8×8 chessboard and can take steps either one square right or one square up.

   (a) How many paths can the chess piece take to get to the top right square?

   (b) How many of those paths avoid the 2×2 center of the chessboard?

3. (AMC 10 2014) Four fair six-sided dice are rolled. What is the probability that at least three of the four dice show the same value?

4. (AIME 2003) An integer between 1000 and 9999, inclusive, is called *balanced* if the sum of its two leftmost digits equals the sum of its two rightmost digits. How many balanced integers are there?

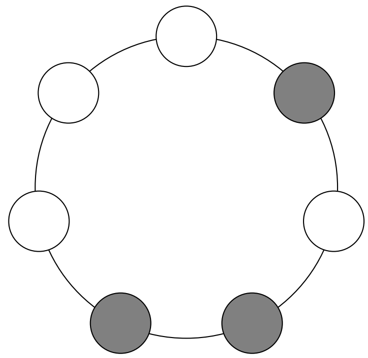
5. (AIME 2002) Count the number of sets $\{A, B\}$, where A and B are *nonempty* subsets of $\{1, 2, 3, \ldots, 10\}$ with no elements in common.

   (For example, one such set is $\{\{3, 6, 7\}, \{2, 9\}\}$. Sets are always unordered.)

6. How many ways are there to rearrange the letters of "FROUFROU" such that no two identical letters are adjacent? ("FROUFROU" itself counts.)

   (If you are ambitious, try "INTESTINES". If you are unambitious, try "POMPOM". If you would like an equivalent problem, try "TEAMMATE" or "HOTSHOTS".)

7. A total of 7 beads of two colors are used to make a bracelet:

   How many different bracelets can be made? (Bracelets can be rotated or flipped over.)

8. (AIME 1996) Five teams play each other in a round-robin tournament; each game is random, with either team having a 50% probability of winning (there are no draws). What is the probability that every team will win at least once, but no team will be undefeated?

9. (AIME 2001) The squares in a 3×3 grid are colored red and blue at random, and each color is equally likely.

   What is the probability that a 2×2 square will be entirely red?

10. (Putnam 1985) Find the number of ordered triples (A_1, A_2, A_3) of sets with the property that

    (i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and

    (ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.