POWER ROUND: MEDITATIONS ON PARTITIONS

ARML 1998

(1) Let positive integers A, B, and C be the angles of a triangle (in degrees) such that $A \leq B \leq C$.

  (a) Determine all the values that each of A, B, and C can take on.

  (b) Compute the number of ordered triples (A, B, C) in which $B = 70°$.

(2) In convex pentagon $ABCDE$, $m\angle A < m\angle B < m\angle C < m\angle D < m\angle E$. Let $T = m\angle C + m\angle D$. If $m\angle A : m\angle B : m\angle C : m\angle D : m\angle E = 1 : 2 : x : y : 5$, determine the range of values of T.

(3) Let a, b, and c be positive integers such that $a < 3b$ and $b > 4c$ and $a + b + c = 200$.

  (a) Determine the largest value that c can take on.

  (b) Determine the smallest value that b can take on.

  (c) Determine the number of ordered triples (a, b, c) in which $c = 11$.

(4) Let a, b, and c be positive integers. If $a + b + c = 85$, $c > 3a$, $2b > c$, and $5a > 3b$, prove algebraically that there is a unique solution (a, b, c) to this system.

(5) A unit square is divided into 4 rectangles of positive area by two cuts parallel to the sides of the square. Let $a_1 \leq a_2 \leq a_3 \leq a_4$ be the areas of the four parts in nondecreasing order. For each $i = 1, \ldots, 4$, determine with proof the range of values for a_i.

(6) A unit cube is divided into 8 parallelepipeds of positive volume by three cuts parallel to the faces of the cube. Let $v_1 \leq v_2 \leq \cdots \leq v_8$ be the volumes of the eight parts in nondecreasing order. Determine with proof the range of values for v_4 and v_5.

(7) Let n be a positive integer. Allie and Bob play a game constructing a partition $n = a_1 + a_2 + \cdots + a_k$ with $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$. Allie wins if there is an odd number of terms in the partition, i.e. if k is odd, and Bob wins otherwise. Allie begins by choosing an a_1 between 1 and $n-1$ inclusive. Bob then chooses an a_2 between 1 and a_1 inclusive such that $a_1 + a_2 \leq n$. Allie then chooses an a_3 between 1 and a_2 inclusive such that $a_1 + a_2 + a_3 \leq n$, and so on, with the game ending when the partition is complete. Determine with proof all $n > 1$ for which Bob has a winning strategy.

(8) Allie and Bob play a game similar to the one in (7) except that the inequality $a_i \geq a_{i+1}$ is replaced by $2a_i \geq a_{i+1}$. Prove that Bob has a winning strategy if and only if n is a Fibonacci number. (You may assume the following: each positive integer n can be uniquely represented as a decreasing sum of non-adjacent Fibonacci numbers, e.g., $32 = 21 + 8 + 3$.)