Complex Numbers Practice

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Problems

1. (2009 AIME I Problem 2) There is a complex number z with imaginary part 164 and a positive integer n such that
$$\frac{z}{z+n} = 4i.$$
Find n.

2. (1985 AIME Problem 3) Find c if a, b, and c are positive integers which satisfy $c = (a+bi)^3 - 107i$, where $i^2 = -1$.

3. (1995 AIME Problem 5) For certain real values of a, b, c, and d, the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four non-real roots. The product of two of these roots is $13+i$ and the sum of the other two roots is $3+4i$, where $i = \sqrt{-1}$. Find b.

4. (1984 AIME Problem 8) The equation $z^6 + z^3 + 1 = 0$ has complex roots with argument θ between $90°$ and $180°$ in the complex plane. Determine the degree measure of θ.

5. (1994 AIME Problem 8) The points $(0,0)$, $(a,11)$, and $(b,37)$ are the vertices of an equilateral triangle. Find the value of ab.

6. (1999 AIME Problem 9) A function f is defined on the complex numbers by $f(z) = (a+bi)z$, where a and b are positive numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|a+bi| = 8$ and that $b^2 = m/n$, where m and n are relatively prime positive integers, find $m+n$.

7. (2000 AIME II Problem 9) Given that z is a complex number such that $z + \frac{1}{z} = 2\cos 3°$, find the least integer that is greater than $z^{2000} + \frac{1}{z^{2000}}$.

8. (2005 AIME II Problem 9) For how many positive integers $n \leq 1000$ is $(\sin t + i\cos t)^n = \sin nt + i\cos nt$ true for all real t?

9. (1990 AIME Problem 10) The sets $A = \{z : z^{18} = 1\}$ and $B = \{w : w^{48} = 1\}$ are both sets of complex roots of unity. The set $C = \{zw : z \in A \text{ and } w \in B\}$ is also a set of complex roots of unity. How many distinct elements are in C?

10. (1992 AIME Problem 10) Consider the region A in the complex plane that consists of all points z such that both $\frac{z}{40}$ and $\frac{40}{\overline{z}}$ have real and imaginary parts between 0 and 1, inclusive. What is the integer that is nearest the area of A?

11. (1988 AIME Problem 11) Let $w_1, w_2, \dots, w_n$ be complex numbers. A line L in the complex plane is called a mean line for the points $w_1, w_2, \dots, w_n$ if L contains points (complex numbers) $z_1, z_2, \dots, z_n$ such that

$$\sum_{k=1}^{n}(z_k - w_k) = 0.$$

For the numbers $w_1 = 32 + 170i$, $w_2 = -7 + 64i$, $w_3 = -9 + 200i$, $w_4 = 1 + 27i$, and $w_5 = -14 + 43i$, there is a unique mean line with y-intercept 3. Find the slope of this mean line.

12. (1996 AIME Problem 11) Let P be the product of the roots of $z^6 + z^4 + z^3 + z^2 + 1 = 0$ that have a positive imaginary part, and suppose that $\mathrm{P} = r(\cos\theta^\circ + i\sin\theta^\circ)$, where $0 < r$ and $0 \le \theta < 360$. Find θ.

13. (1997 AIME Problem 11)

    Let $x = \dfrac{\sum\limits_{n=1}^{44} \cos n^\circ}{\sum\limits_{n=1}^{44} \sin n^\circ}$. What is the greatest integer that does not exceed $100x$?

14. (2002 AIME I Problem 12) Let $F(z) = \frac{z+i}{z-i}$ for all complex numbers $z \neq i$, and let $z_n = F(z_{n-1})$ for all positive integers n. Given that $z_0 = \frac{1}{137} + i$ and $z_{2002} = a + bi$, where a and b are real numbers, find $a + b$.

15. (2004 AIME I Problem 13) The polynomial $P(x) = (1 + x + x^2 + \cdots + x^{17})^2 - x^{17}$ has 34 complex roots of the form $z_k = r_k[\cos(2\pi a_k) + i\sin(2\pi a_k)], k = 1, 2, 3, \dots, 34$, with $0 < a_1 \le a_2 \le a_3 \le \cdots \le a_{34} < 1$ and $r_k > 0$. Given that $a_1 + a_2 + a_3 + a_4 + a_5 = m/n$, where m and n are relatively prime positive integers, find $m + n$.

16. (1994 AIME Problem 13) The equation $x^{10} + (13x - 1)^{10} = 0$ has 10 complex roots $r_1, \overline{r_1}, r_2, \overline{r_2}, r_3, \overline{r_3}, r_4, \overline{r_4}, r_5, \overline{r_5}$, where the bar denotes complex conjugation. Find the value of

$$\frac{1}{r_1\overline{r_1}} + \frac{1}{r_2\overline{r_2}} + \frac{1}{r_3\overline{r_3}} + \frac{1}{r_4\overline{r_4}} + \frac{1}{r_5\overline{r_5}}$$

17. (1998 AIME Problem 13) If $\{a_1, a_2, a_3, \dots, a_n\}$ is a set of real numbers, indexed so that $a_1 < a_2 < a_3 < \cdots < a_n$, its complex power sum is defined to be $a_1 i + a_2 i^2 + a_3 i^3 + \cdots + a_n i^n$, where $i^2 = -1$. Let S_n be the sum of the complex power sums of all nonempty subsets of $\{1, 2, \dots, n\}$. Given that $S_8 = -176 - 64i$ and $S_9 = p + qi$, where p and q are integers, find $|p| + |q|$.

18. (1989 AIME Problem 14) Given a positive integer n, it can be shown that every complex number of the form $r + si$, where r and s are integers, can be uniquely expressed in the base $-n + i$ using the integers $1, 2, \dots, n^2$ as digits. That is, the equation

$$r + si = a_m(-n + i)^m + a_{m-1}(-n + i)^{m-1} + \cdots + a_1(-n + i) + a_0$$

is true for a unique choice of a non-negative integer m and digits $a_0, a_1, \dots, a_m$ chosen from the set $\{0, 1, 2, \dots, n^2\}$, with $a_m \neq 0$. We write

$$r + si = (a_m a_{m-1} \dots a_1 a_0)_{-n+i}$$

to denote the base $-n + i$ expansion of $r + si$. There are only finitely many integers $k + 0i$ that have four-digit expansions

$$k = (a_3 a_2 a_1 a_0)_{-3+i}, \quad a_3 \neq 0$$

Find the sum of all such k.