Mock ARML: Team Round

ARML Practice 10/12/2014

1. If $A, B, C, \ldots, J$ are distinct digits, compute the number of possible values that the sum of two-digit numbers
$$AB + CD + EF + GH + IJ$$
can take on. *(Leading digits cannot be 0.)*

2. Define $\log_2^* n$ to be the least integer k such that $\underbrace{\log_2 \log_2 \cdots \log_2}_{k} n \leq 1$. For example, $\log_2^* 8 = 3$ since $\log_2 \log_2 8 = \log_2 3 \approx 1.58$ but $\log_2 \log_2 \log_2 8 = \log_2 \log_2 3 \approx 0.66$.

   Compute $\log_2^* 1000!$.

3. An integer N is worth 1 point for each adjacent pair of digits that form a perfect square of a positive integer. For example, 3604 is worth 2 points, because 36 and 04 are perfect squares, but 60 is not. Compute the smallest positive integer that is worth 5 points.

4. Each side and diagonal of regular hexagon $A_1A_2A_3A_4A_5A_6$ is colored either red or blue. Compute the smallest possible number of triangles $A_iA_jA_k$, all three of whose sides are the same color.

5. Let n be the smallest positive integer of the form $2^a 5^b$ which has at least 100 positive divisors. Compute the ordered pair (a, b).

6. Given an arbitrary finite sequence of letters (represented as a word), a *subsequence* is a sequence of one or more letters that appear in the same order as in the original sequence. For example, N, CT, OTT, and $CONTEST$ are subsequence of the word $CONTEST$, but NOT, $ONSET$, and $TESS$ are not. Assuming the standard English alphabet $\{A, B, \ldots, Z\}$, compute the number of distinct five-letter sequences which have $MATH$ as a subsequence.

7. Five four-dimensional hyperspheres of radius 1 are pairwise externally tangent (which means that any two of the hyperspheres meet at a single point, and their interiors are disjoint). They are also each internally tangent to a hypersphere of radius $r > 1$ (which means that each meets the larger hypersphere at a single point, and their interiors are contained in the interior of the larger hypersphere). Compute r.

8. The equations $x^3 + x^2 + Ax + 10 = 0$ and $x^3 - x^2 + Bx + 50 = 0$ have two roots in common. Compute the sum of these common roots.

9. A sequence x_n is defined by $x_1 = a$, $x_2 = b$, and $x_n = x_{n-1} + x_{n-2}$ for $n \geq 3$, where a and b are integers and $1 \leq a, b \leq 9$. Compute the number of possible values of a and b for which the sequence x_n contains the term 10.

10. In $\triangle ABC$ with side lengths 3, 4, and 5, angle bisectors AX, BY, and CZ are drawn, where X lies on BC, Y lies on AC, and Z lies on AB. Compute the area of $\triangle XYZ$.