Mock ARML

Individual Round

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ARML Practice 10/12/2014

Problems 1 and 2

1. Compute the next year after 2014 which cannot be written as the difference of two squares of integers.

2. Five congruent circles of radius r are drawn in square $ABCD$: one tangent to AB and BC, one tangent to BC and CD, one tangent to CD and AD, one tangent to AD and AB, and one tangent to the other four. If $AB = 1$, compute r.

Problems 1 and 2

1. Compute the next year after 2014 which cannot be written as the difference of two squares of integers.

Answer: 2018

2. Five congruent circles of radius r are drawn in square $ABCD$: one tangent to AB and BC, one tangent to BC and CD, one tangent to CD and AD, one tangent to AD and AB, and one tangent to the other four. If $AB = 1$, compute r.

Answer: $\mathbf{\dfrac{\sqrt{2}-1}{2}}$

Problems 3 and 4

3. Let A, B, and C be digits, $A \neq 0$. If $AB \cdot AC = 1023$, compute $AB + AC$.

4. $T(x)$ is a degree 3 polynomial with integer coefficients, one of whose roots is $\cos \frac{\pi}{7}$. Compute $\frac{T(1)}{T(-1)}$.

Problems 3 and 4

3. Let A, B, and C be digits, $A \neq 0$. If $AB \cdot AC = 1023$, compute $AB + AC$.

Answer: 64

4. $T(x)$ is a degree 3 polynomial with integer coefficients, one of whose roots is $\cos \frac{\pi}{7}$. Compute $\frac{T(1)}{T(-1)}$.

Answer: $-\frac{1}{7}$

Problems 5 and 6

5. A regular 12-gon has vertices $A_1, A_2, \ldots, A_{12}$. If A_3, A_6, A_9, and A_{12} are the midpoints of the four sides of a square with area 1, compute the area of the 12-gon.

6. Compute all integer values of k for which $x = \cos\alpha$ is a solution to $x^2 + x - k = 0$ for some real α.

Problems 5 and 6

5. A regular 12-gon has vertices $A_1, A_2, \ldots, A_{12}$. If A_3, A_6, A_9, and A_{12} are the midpoints of the four sides of a square with area 1, compute the area of the 12-gon.

Answer: $\frac{\mathbf{3}}{\mathbf{4}}$

6. Compute all integer values of k for which $x = \cos\alpha$ is a solution to $x^2 + x - k = 0$ for some real α.

Answer: $\mathbf{k = 0, 1, 2}$

Problems 7 and 8

7. Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be two geometric sequences. The sequence $c_1, c_2, c_3, \ldots$ has $c_n = a_n + b_n$ for each positive integer n. If $c_1 = 1$, $c_2 = 4$, $c_3 = 3$, and $c_4 = 2$, compute c_5.

8. In triangle ABC, $AB = 3$, $BC = 4$, and $AC = 5$. $\triangle ABC$ is inscribed in a circle, and D is the point on the circle opposite B. Compute the distance between the incenters of $\triangle ABC$ and $\triangle ACD$.

Problems 7 and 8

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Answer: $\frac{17}{13}$

8. In triangle ABC, $AB = 3$, $BC = 4$, and $AC = 5$. $\triangle ABC$ is inscribed in a circle, and D is the point on the circle opposite B. Compute the distance between the incenters of $\triangle ABC$ and $\triangle ACD$.

Answer: $\sqrt{5}$

Problems 9 and 10

9. Diameters AB and CD of circle S are perpendicular. Chord AE intersects diameter CD at point K and chord EC intersects diameter AB at point L. If $CK : KD = 2 : 1$, find $AL : LB$.

10. Compute the largest integer between 1 and 1000 whose base 5 representation consists of the last k digits of its base 2 representation, for some k.

Problems 9 and 10

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Answer: 3 : 1

10. Compute the largest integer between 1 and 1000 whose base 5 representation consists of the last k digits of its base 2 representation, for some k.

Answer: 31