Warm-up

1. (ARML 2007) In rectangle $ABCD$, $M$ is the midpoint of $AB$, $AC$ and $DM$ intersect at $E$, $CE = 10$, and $EM = 4$. Find the area of rectangle $ABCD$.

Problems

1. (ARML 1993) Triangle $AOB$ is positioned in the first quadrant with $O = (0,0)$ and $B$ above and to the right of $A$. The slope of $OA$ is 1, the slope of $OB$ is 8, and the slope of $AB$ is $m$. If the points $A$ and $B$ have $x$-coordinates $a$ and $b$, respectively, compute $\frac{b}{a}$ in terms of $m$.

2. (ARML 1993) Square $ABCD$ is positioned in the first quadrant with $A$ on the $y$-axis, $B$ on the $x$-axis, and $C = (13, 8)$. Compute the area of the square.

3. (AIME 2000) Let $u$ and $v$ be integers satisfying $0 < v < u$. Let $A = (u, v)$, let $B$ be the reflection of $A$ across the line $y = x$, let $C$ be the reflection of $B$ across the $y$-axis, let $D$ be the reflection of $C$ across the $x$-axis, and let $E$ be the reflection of $D$ across the $y$-axis. The area of pentagon $ABCDE$ is 451. Find $u + v$.

4. (AIME 2001) Let $R = (8, 6)$. The lines whose equations are $8y = 15x$ and $10y = 3x$ contain points $P$ and $Q$, respectively, such that $R$ is the midpoint of $PQ$. The length of $PQ$ equals $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

5. (ARML 1988 Power Round)

(a) A sequence $(x_n)$ is defined as follows: $x_0 = 2$, and for all $n \geq 1$, $(x_n, 0)$ lies on the line through $(0,4)$ and $(x_{n-1}, 2)$. Derive a formula for $x_n$ in terms of $x_{n-1}$.

(b) A sequence $(y_n)$ is defined as follows: $y_0 = 0$, and for all $n \geq 1$, draw a square of side length 2 with its bottom left corner at $(y_{n-1}, 0)$ and its bottom side on the $x$-axis. The point $(y_n, 0)$ lies on the line through $(0,4)$ and the top right corner of the square. Derive a formula for $y_n$ in terms of $y_{n-1}$.

(c) A sequence $(z_n)$ is defined as follows: $z_0 = 0$, and for all $n \geq 1$, draw a circle of diameter 2 tangent to the $x$-axis and tangent to the line through $(0,4)$ and $(z_{n-1}, 0)$ in such a
way that its center lies to the right of that line. The line through (0, 4) and (zn, 0) is the other tangent to the same circle. Derive a formula for zn in terms of zn−1.

(d) Express (xn), (yn), and (zn) explicitly as functions of n.

6. Prove that the area of a triangle with coordinates (a, b), (c, d), and (e, f) is given by

\[ \frac{1}{2} \left| \det \begin{pmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{pmatrix} \right| = \frac{1}{2} |ad + be + cf - af - bc - de| . \]

7. (AIME 2005) The points A = (p, q), B = (12, 19), and C = (23, 20) form a triangle of area 70. The median from A to side BC has slope −5. Find the largest possible value of p + q.

8. (a) Prove that the medians of a triangle can be translated (without rotating the line segments) to form the sides of a new triangle.

(b) The medians of \( \triangle ABC \) are translated to form the sides of \( \triangle DEF \), and the medians of \( \triangle DEF \) are translated to form the sides of \( \triangle GHI \). Prove that \( \triangle ABC \) and \( \triangle GHI \) are similar, and compute the coefficient of similarity.

9. (ARML 2001) Let \( ABCDEFGH \) be a rectangular box such that \( AB = AD = 20 \) and \( \angle GAC = 45^\circ \). Point P lies on DH such that plane PAC is parallel to BH. Compute the volume of tetrahedron FPCA.

10. (a) A sphere of radius \( r \) is inscribed in a regular tetrahedron, and a sphere of radius \( R \) is circumscribed about the same tetrahedron. Find the ratio \( R : r \).

(b) An \( n \)-dimensional sphere of radius \( r \) is inscribed in a regular \( n \)-dimensional simplex (a figure with \( n + 1 \) vertices and \( n + 1 \) faces which are all regular \( (n - 1) \)-dimensional simplices; in 1, 2, and 3 dimensions a simplex is a line segment, triangle, and tetrahedron respectively), and an \( n \)-dimensional sphere of radius \( R \) is circumscribed about the same simplex. Find the ratio \( R : r \).

11. Find the equation of the line that bisects the angle formed in the first quadrant by the x-axis and the line \( y = mx \).

12. In \( \triangle ABC \), the altitude \( AH \) and the median \( AM \) are drawn; points H and M are distinct and points B, H, M, and C are in that order on segment BC. If \( \angle BAH = \angle MAC \), compute \( \angle BAC \).