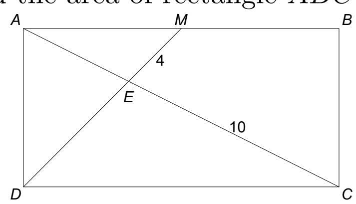
Geometry	Misha Lavrov

Coordinate Geometry

Western PA ARML Practice *October 29, 2014*

Warm-up

1. (ARML 2007) In rectangle $ABCD$, M is the midpoint of AB, AC and DM intersect at E, $CE = 10$, and $EM = 4$. Find the area of rectangle $ABCD$.



Problems

1. (ARML 1993) Triangle AOB is positioned in the first quadrant with $O = (0, 0)$ and B above and to the right of A. The slope of OA is 1, the slope of OB is 8, and the slope of AB is m. If the points A and B have x-coordinates a and b, respectively, compute $\frac{b}{a}$ in terms of m.

2. (ARML 1993) Square $ABCD$ is positioned in the first quadrant with A on the y-axis, B on the x-axis, and $C = (13, 8)$. Compute the area of the square.

3. (AIME 2000) Let u and v be integers satisfying $0 < v < u$. Let $A = (u, v)$, let B be the reflection of A across the line $y = x$, let C be the reflection of B across the y-axis, let D be the reflection of C across the x-axis, and let E be the reflection of D across the y-axis. The area of pentagon $ABCDE$ is 451. Find $u + v$.

4. (AIME 2001) Let $R = (8, 6)$. The lines whose equations are $8y = 15x$ and $10y = 3x$ contain points P and Q, respectively, such that R is the midpoint of PQ. The length of PQ equals $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

5. (ARML 1988 Power Round)

   (a) A sequence (x_n) is defined as follows: $x_0 = 2$, and for all $n \geq 1$, $(x_n, 0)$ lies on the line through $(0, 4)$ and $(x_{n-1}, 2)$. Derive a formula for x_n in terms of x_{n-1}.

   (b) A sequence (y_n) is defined as follows: $y_0 = 0$, and for all $n \geq 1$, draw a square of side length 2 with its bottom left corner at $(y_{n-1}, 0)$ and its bottom side on the x-axis. The point $(y_n, 0)$ lies on the line through $(0, 4)$ and the top right corner of the square. Derive a formula for y_n in terms of y_{n-1}.

   (c) A sequence (z_n) is defined as follows: $z_0 = 0$, and for all $n \geq 1$, draw a circle of diameter 2 tangent to the x-axis and tangent to the line through $(0, 4)$ and $(z_{n-1}, 0)$ in such a way that its center lies to the right of that line. The line through $(0,4)$ and $(z_n, 0)$ is the other tangent to the same circle. Derive a formula for z_n in terms of z_{n-1}.

(d) Express (x_n), (y_n), and (z_n) explicitly as functions of n.

6. Prove that the area of a triangle with coordinates (a,b), (c,d), and (e,f) is given by

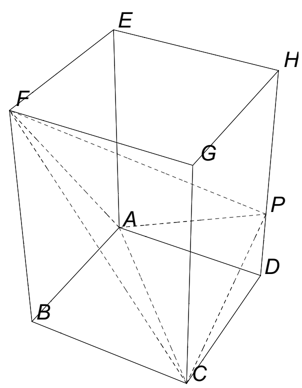
$$\frac{1}{2}\left|\det\begin{pmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{pmatrix}\right| = \frac{1}{2}\left|ad + be + cf - af - bc - de\right|.$$

7. (AIME 2005) The points $A = (p,q)$, $B = (12,19)$, and $C = (23,20)$ form a triangle of area 70. The median from A to side BC has slope -5. Find the largest possible value of $p+q$.

8. (a) Prove that the medians of a triangle can be translated (without rotating the line segments) to form the sides of a new triangle.

   (b) The medians of $\triangle ABC$ are translated to form the sides of $\triangle DEF$, and the medians of $\triangle DEF$ are translated to form the sides of $\triangle GHI$. Prove that $\triangle ABC$ and $\triangle GHI$ are similar, and compute the coefficient of similarity.

9. (ARML 2001) Let $ABCDEFGH$ be a rectangular box such that $AB = AD = 20$ and $\angle GAC = 45°$. Point P lies on DH such that plane PAC is parallel to BH. Compute the volume of tetrahedron $FPCA$.



10. (a) A sphere of radius r is inscribed in a regular tetrahedron, and a sphere of radius R is circumscribed about the same tetrahedron. Find the ratio $R : r$.

    (b) An n-dimensional sphere of radius r is inscribed in a regular n-dimensional simplex (a figure with $n+1$ vertices and $n+1$ faces which are all regular $(n-1)$-dimensional simplices; in 1, 2, and 3 dimensions a simplex is a line segment, triangle, and tetrahedron respectively), and an n-dimensional sphere of radius R is circumscribed about the same simplex. Find the ratio $R : r$.

11. Find the equation of the line that bisects the angle formed in the first quadrant by the x-axis and the line $y = mx$.

12. In $\triangle ABC$, the altitude AH and the median AM are drawn; points H and M are distinct and points B, H, M, and C are in that order on segment BC. If $\angle BAH = \angle MAC$, compute $\angle BAC$.