Number Theory

ARML Homework #2

October 6, 2013

Homework disclaimer: I believe that math is interesting, and I try to think of problems that I think are interesting. If you think they're interesting too, then you will want to do them. I'll leave it at that. If you solve some problems and want feedback on your answers, email them to me at `mlavrov@andrew.cmu.edu`. (For problems that aren't proofs, my feedback will be much more useful if you show work.)

1. Go to the PUMaC website and find a copy of the competition from a previous year. (Doing this is an important, though not math-related, part of the problem.)

   Do all of the number theory problems.

2. Here is a silly number theory parlor trick.

   Ask your victim to pick a number x between 1 and 60. Have them tell you $x \bmod 3$, $x \bmod 4$, and $x \bmod 5$. Then compute $40(x \bmod 3) - 15(x \bmod 4) + 36(x \bmod 5)$ and take the result mod 60. Triumphantly tell your victim that this was the number they chose.

   Explain why this trick works.

3. Suppose you want a variant of the trick in problem 2, where your victim picks a 3-digit number x, and tells you $x \bmod 7$, $x \bmod 11$, and $x \bmod 13$. Find a formula to get back x.

4. Let p be a prime. We say that "x is a $\sqrt{-1}$ for p" to mean that $x^2 \equiv -1 \pmod{p}$.

   (a) Which primes p have a $\sqrt{-1}$? When they do, how many do they have?

   (b) Let p be a prime that has a $\sqrt{-1}$, and let i be a particular choice of a $\sqrt{-1}$ for p. Prove that if $a^2 + b^2$ is divisible by p, then either $a + bi$ or $a - bi$ is divisible by p.

5. Find a polynomial $p(x)$ with integer coefficients such that the coefficients of x are not all $0 \bmod 10$, but for all integer x, $p(x) \equiv 0 \pmod{10}$.