Number Theory

Miscellaneous tricks

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ARML Practice 5/19/2013

Warm-up

Problem (ARML 1995/I3.)

Find all primes p such that $p^{1994} + p^{1995}$ is a perfect square.

Problem (PUMaC 2009 Number Theory.)

If $17! = 355687ab8096000$, where a and b are two missing digits, find a and b.

Problem (ARML 1995/T10.)

Let $x = \log_{10} 14$, $y = \log_{10} 15$, $z = \log_{10} 16$. How many elements of the set

$$\{\log_{10} 1, \log_{10} 2, \ldots, \log_{10} 100\}$$

can be written as $ax + by + cz + d$ for some rational numbers a, b, c, and d?

Warm-up

Solutions, sort of

- Since p^{1994} is always a perfect square, we must also have $p+1$ be a perfect square. So $p = k^2 - 1 = (k+1)(k-1)$, which can only hold if $k = 2$ and $p = 3$.

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- 17! is divisible both by 9 and by 11, so:
  - $3 + 5 + \cdots + a + b + \cdots \equiv 0 \pmod 9$, so $a + b \equiv 6 \pmod 9$.
  - $3 - 5 + \cdots + a - b - \cdots \equiv 0 \pmod{11}$, so $a - b \equiv 2 \pmod{11}$.

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  This means $a = 4$ and $b = 2$.
- Using x, y, z, and 1, we can write the logs of 2, 3, 5, and 7. So anything we can make by multiplying 2's, 3's, 5's, and 7's is fair game. There's 46 of those.

The grid of divisors

Problem (AIME 1988/5.)

Find the probability that a positive integer divisor of 10^{99}*, chosen uniformly at random, is an integer multiple of* 10^{88}*.*

Problem (AIME 1995/6.)

Let $n = 2^{31}3^{19}$*. How many positive integer divisors of* n^2 *are less than* n *but do not divide* n*?*

Solution

AIME 1988/5

The divisors of 10^{99} can be arranged in a 100×100 square grid that looks like this:

$$\begin{array}{ccccc} 1 & 2 & 4 & \cdots & 2^{99} \\ 5 & 10 & 20 & \cdots & 2^{99} \cdot 5 \\ 25 & 50 & 100 & \cdots & 2^{99} \cdot 5^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 5^{99} & 2 \cdot 5^{99} & 4 \cdot 5^{99} & \cdots & 10^{99} \end{array}$$

(The picture is never actually necessary, but is good to keep in your head anyway.)

The integer multiples of $10^{88} = 2^{88} \cdot 5^{88}$ form a 12×12 sub-square of this grid (its corners are at $2^{88} \cdot 5^{88}$, $2^{88} \cdot 5^{99}$, $2^{99} \cdot 5^{99}$, and $2^{99} \cdot 5^{88}$). So the probability is $\frac{12^2}{100^2} = \frac{9}{625}$.

Solution

AIME 1995/6.

There are three parts to this:

1. How many divisors does n^2 have?

2. How many divisors of n^2 are less than n?

3. How many these are actually divisors of n?

Solution

AIME 1995/6.

There are three parts to this:

1. How many divisors does n^2 have?

   Since $n^2 = 2^{62} \cdot 3^{38}$, there are 63 choices for the power of 2 and 39 for the power of 3, so there are $63 \cdot 39$ divisors.

2. How many divisors of n^2 are less than n?

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2. How many divisors of n^2 are less than n?

   If d is a divisor, so is $\frac{n^2}{d}$, and unless $d = n$ exactly one of these is less than n, so $\frac{63 \cdot 39 - 1}{2}$ are less than n.

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   As in step 1, we have 32 choices for the power of 2 and 20 for the power of 3, so there are $32 \cdot 20 - 1$ divisors not counting n.

Solution

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So our final answer is $\frac{63 \cdot 39 - 1}{2} - (32 \cdot 20 - 1) = 589$.

The grid of divisors

Problem

Let $n = 30^4$, which has 125 divisors. How many divisors of n have, themselves, an odd number of divisors?
(Extension: how would you compute this for an arbitrary n?)

Problem (AIME 1996/8.)

The harmonic mean of x and y is the reciprocal of the mean of their reciprocals:

$$\mathrm{HM}(x, y) = \frac{1}{\frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right)}.$$

How many pairs of positive integers x, y have harmonic mean 6^{20}?

Solution

odd number of divisors

If $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$, then there are $(a_1+1)(a_2+1)(\cdots)(a_k+1)$ divisors: when choosing the power of p_i to use, we have $a_i + 1$ choices from 0 to a_i. This is odd if $a_1, \dots, a_k$ are all even.

In the problem for $n = 30^4 = 2^4 \cdot 3^4 \cdot 5^4$, a divisor of n with an odd number of divisors must have 0, 2, or 4 factors of each of 2, 3, and 5. There are 27 such choices.

To solve this in general, for each prime p_i we choose one of $0, 2, \dots, 2\left\lfloor \frac{a_i}{2} \right\rfloor$ as the number of times p_i is used, so the number of such divisors is

$$\left(\left\lfloor \frac{a_1}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{a_2}{2} \right\rfloor + 1\right)\cdots\left(\left\lfloor \frac{a_k}{2} \right\rfloor + 1\right).$$

Solution
AIME 1996/8.

We can write

$$\mathrm{HM}(x, y) = \frac{1}{\frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right)} = \frac{2xy}{x + y}.$$

If $\mathrm{HM}(x, y) = 6^{20} = 2^{20}3^{20}$, then

$$2xy = 6^{20}(x + y)$$

which factors as

$$(x - 2^{19}3^{20})(y - 2^{19}3^{20}) = 2^{38}3^{40}.$$

So we just count the factors of $2^{38}3^{40}$.

Shuffling prime numbers around

Problem (AIME 1991/5.)

How many rational numbers are there between 0 *and* 1 *which, when written as* $\frac{a}{b}$ *for relatively prime* a *and* b, *satisfy* $ab = 20!$?

Problem (PUMaC 2009/NTB 4.)

How many positive integer pairs (a, b) *satisfy*

$$a^2 + b^2 = ab(a + b)?$$

Solution
AIME 1991/5

For each prime p that divides 20!, it must divide either only a or only b, because $\frac{a}{b}$ is reduced. So we have two choices for what to do with p: we could put it all in a, or all in b.

There are 8 prime factors of 20!: 2, 3, 5, 7, 11, 13, 17, and 19. So it seems like we have $2^8 = 256$ choices.

However, we wanted a fraction $\frac{a}{b}$ between 0 and 1, meaning $a < b$. This happens for exactly half of our choices, so the answer is not 256 but 128.

Solution

PUMaC 2009/NTB 4

We can rewrite the equation as

$$(a+b)^2 = ab(a+b+2).$$

We play the prime factor game. Suppose p divides a. Then it divides the RHS, so it divides the LHS, so it divides $a+b$, so it also divides b.

With more precision: suppose the highest powers of p dividing a are m and n. If $m \neq n$, then we have at least p^{m+n} on the right, and at most $p^{2\min(m,n)}$ on the left, which is impossible, so $m = n$. Therefore $a = b$. This is only possible when $a = b = 1$.