Even More Games

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ARML Practice 2/24/2013

Review of basic ideas

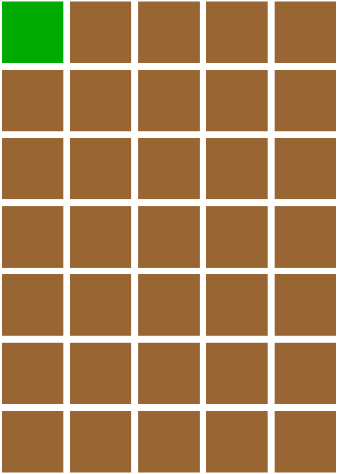
Problem ("Chomp")

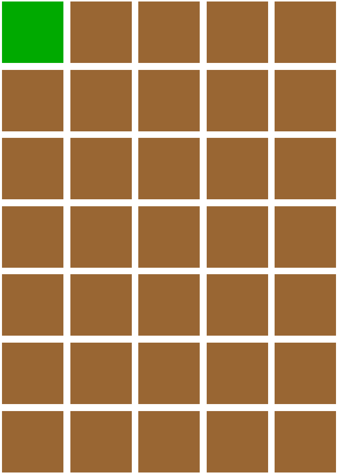
Two players play a game on an $m \times n$ chocolate bar made up of small squares. The players take turns choosing a square and eating it, together with all the squares below it and to the right. The top left square is poisoned: the player who eats it, loses.

1. *Show that the first player has a winning strategy.*
2. *Find this winning strategy in the case $m = n$.*

Solution to Chomp: # 1

This is a strategy-stealing argument:





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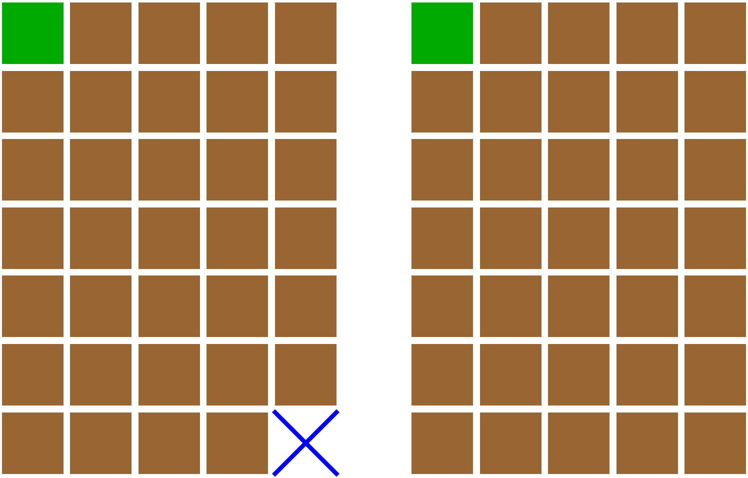

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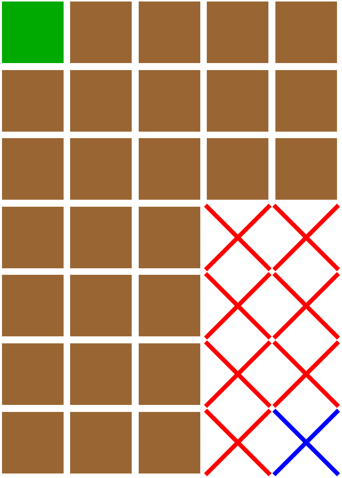

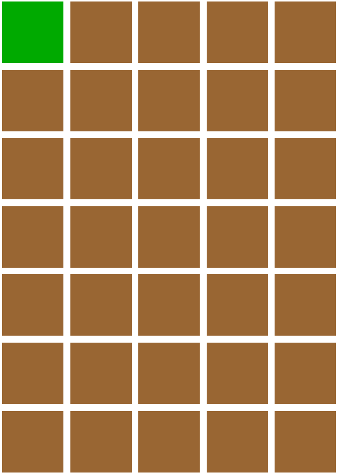

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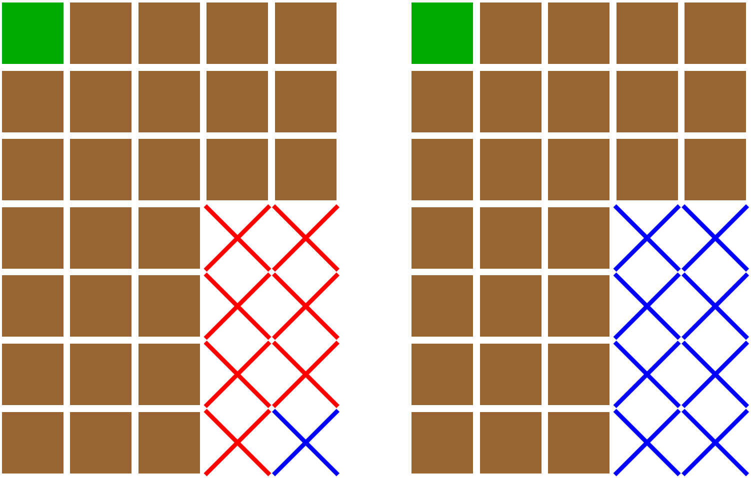
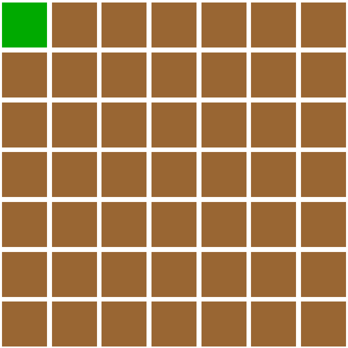


Solution to Chomp: # 2

Start by eating the square below and to the right of the poisoned square. . .



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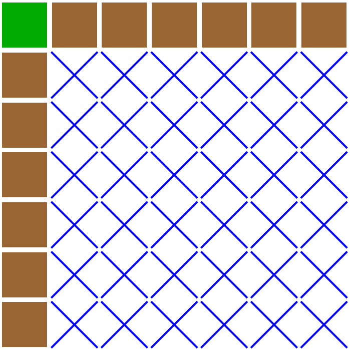

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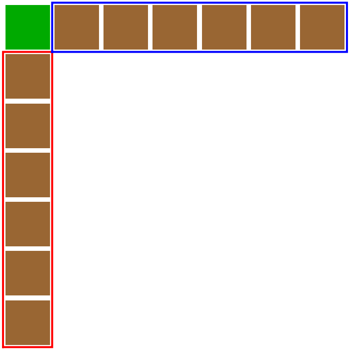

. . . then maintain symmetry between the two long thin pieces.

More about winning positions

Problem (Unknown source)

A box contains 300 *matches. Two players take turns taking some matches from the box; each player must take at least one match, but no more than half the matches. The player who cannot move, loses. Who has the winning strategy?*

Problem (German Math Olympiad 1984/1.)

Two players take turns writing a number $1, 2, \ldots, 6$ *on the board. When* $2n$ *numbers have been written, the game ends; the second player wins if the sum of the numbers is divisible by* 9*. For which values of* n *does the second player have a winning strategy?*

More complicated strategies

Problem (Putnam 1993/B2.)

Cards numbered $1, \ldots, 2n$ *are shuffled and dealt to two players (each receives n cards). The players take turns discarding a card face-up; if the sum of all discarded cards is divisible by* $2n+1$*, the game ends and the player who just discarded, wins.*

Assuming optimal play, who wins and how?

Problem (New Zealand IMO Selection, 2004.)

The numbers $1, \ldots, 1000$ *are written on the board. Two players take turns erasing a number; a number* x *may be erased if* $x = 1$*, or* $x - 1$ *has been erased, or* x *is even and* $\frac{x}{2}$ *has been erased. The player to erase* 1000 *wins.*

Which player has a winning strategy?

Actual ARML problems

Problem (ARML 1998 Power Round)

Allie and Bob play a game constructing a partition

$$n = a_1 + a_2 + \cdots + a_k \quad s.t. \; a_1 \geq a_2 \geq \cdots \geq a_k \geq 1.$$

On the i-th turn, a player picks a_i *such that* $a_i \leq a_{i-1}$ *and* $a_1 + \cdots + a_i \leq n$*. Allie goes first but cannot pick* n*.*

The player to write down a_k *so that* $a_1 + \cdots + a_k = n$*, wins.*

- *For which* n *does Bob have a winning strategy?*
- *Suppose instead of* $a_i \leq a_{i-1}$ *the condition is* $a_i \leq 2a_{i-1}$*. For which* n *does Bob have a winning strategy?*