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Section

# Homework 6

1. First find critical points:

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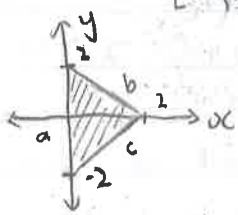
$\vec{\nabla} f = \begin{bmatrix} 2x-2 \\ 2y \end{bmatrix} = \vec{0} \Rightarrow (x,y) = (1,0)$

$f(1,0) = -1$



Along a: $x=0$, $f(0,y) = y^2$ $y \in [-2,2]$

$f(0,2) = f(0,-2) = 4$. $f_y = 2y = 0 \Rightarrow y=0$

Need to consider $f(0,\pm 2)$, $f(0,0)$

Along b: $y = 2-x$. $x \in [0,2]$.

$\cancel{f(0,2)}$, $f(2,0) = 4-2\cdot 2 = 0$.

$f_x = 2x + (-2)(2-x) - 2 = 4x - 6 = 0 \Rightarrow x = 3/2$.

Need to consider $f(2,0)$, $f(3/2, \frac{1}{2})$

Along c: $y = x-2$. $x \in [0,2]$. $\cancel{f(0,-2)}$, $\cancel{f(2,0)}$

$f_x = 2x + 2(x-2) - 2 = 4x - 6 = 0 \Rightarrow x = 3/2$.

Need to consider $f(3/2, \frac{-1}{2})$

$f(1,0) = -1$ (Minimum), $f(0,\pm 2) = 4$ (Maximum), $f(2,0) = 0$

$f(3/2, 1/2) = f(3/2, \frac{-1}{2}) = \frac{-1}{2}$.

The maxima are $(0,\pm 2)$ ✓ and minimum is ✓ $(1,0)$

2. First find critical points

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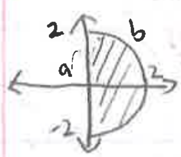
$\vec{\nabla} f = \begin{bmatrix} y^2 \\ 2xy \end{bmatrix} = \vec{0} \Rightarrow y = 0$ is necessary and sufficient

$f(x,0) = 0$



Along a: $x = 0$, $f(0,y) = 0$ along this edge

Along b: $y^2 = 4-x^2$, $f(x,y) = x(4-x^2) = -x^3 + 4x$

$f_x = -3x^2 + 4 = 0 \Rightarrow x = \pm\sqrt{4/3} \overset{x \geq 0}{\Rightarrow} x = \sqrt{4/3}$.

$f(\sqrt{4/3}, \pm\sqrt{8/3}) = \sqrt{4/3} \cdot 8/3 = 3.0792$

The minima are $\{(x,y) \mid (x=0 \wedge y \in [-2,2]) \vee (y=0 \wedge x \in [0,2])\}$

The maxima are $(\sqrt{4/3}, \pm\sqrt{8/3})$

3. If $y_0 = 0$, then $f(x_0, y_0) = x_0^2+1 / x_0^4+1$.

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If $x_0 \in [-1,1]$, then $x_0^2 \leq 1 \Rightarrow x_0^4 \leq x_0^2$

$\Rightarrow x_0^4 + 1 \leq x_0^2 + 1 \Rightarrow f(x_0, 0) \geq 1$.

But, $f(2,0) = 5/17 < 1$, so $x_0 \notin [-1,1]$

So $|x_0| > 1$. But $f(2x_0, 0) = 4x_0^2+1 / 16x_0^4+1$

$\leq 4x_0^2+4 / 16x_0^4 = x_0^2+1 / 4x_0^4 < x_0^2+1 / 3x_0^4+1$ $(x_0^4 > 1)$

Since $x_0^4 > 0$, we have $3x_0^4+1 > x_0^4+1$

So $f(2x_0, 0) < x_0^2+1 / x_0^4+1 = f(x_0, 0)$

$|x_0| \leq 1$ and $|x_0| > 1$ both lead to contradictions.

This forces us to reject the initial $y_0 = 0$ assumption.

But $f(\sqrt{x_0^2+y_0^2}, 0) = x_0^2+y_0^2+1 / (x_0^2+y_0^2)^2+1$

Since $y_0 \neq 0$, we know $y_0^2 > 0$. So $x_0^2+y_0^2+1 < x_0^2+2y_0^2+1$

Also $x_0^2 \geq 0$ so $(x_0^2+y_0^2)^2+1 = x_0^4+y_0^4+2x_0^2y_0^2+1$

$\geq x_0^4+y_0^4+1$

So $f(\sqrt{x_0^2+y_0^2}, 0) < f(x_0, y_0)$ if $y_0 \neq 0$.

We reach a contradiction on all branches, so the function cannot have a minimum.

4. $\vec{\nabla} \times \vec{h} = $ " $\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} y^2 \\ -x \\ y \end{pmatrix}$ " $= \begin{pmatrix} \partial/\partial y\, y + \partial/\partial z\, x \\ \partial/\partial z\, y^2 - \partial/\partial x\, y \\ -\partial/\partial x\, x - \partial/\partial y\, y^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1-2y \end{pmatrix} \neq \vec{0}$.

If $\exists f: \mathbb{E}^3 \to \mathbb{E}$ $\vec{\nabla} f = \vec{h}$, then $\vec{\nabla} \times \vec{h} = \vec{\nabla} \times \vec{\nabla} f = \vec{0}$.

This is a contradiction, so no such $f$ exists.

5. If $\vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k}$ and $\vec{w} = w_x \vec{i} + w_y \vec{j} + w_z \vec{k}$

then $\vec{\nabla} \cdot (\vec{v} \times \vec{w}) = \partial/\partial x (v_y w_z - v_z w_y) + \partial/\partial y (v_z w_x - v_x w_z)$

$+ \partial/\partial z (v_x w_y - v_y w_x)$

$= w_x (\partial/\partial y\, v_z - \partial/\partial z\, v_y) + w_y (\partial/\partial z\, v_x - \partial/\partial x\, v_z)$

$+ w_z (\partial/\partial x\, v_y - \partial/\partial y\, v_x) + v_x (-\partial/\partial y\, w_z + \partial/\partial z\, w_y)$

$+ v_y (-\partial/\partial z\, w_x + \partial/\partial x\, w_z) + v_z (-\partial/\partial x\, w_y + \partial/\partial y\, w_x)$

$= \vec{w} \cdot (\vec{\nabla} \times \vec{v}) - \vec{v} \cdot (\vec{\nabla} \times \vec{w})$

$\forall (x,y)$:

6. $0 \leq \frac{3x^2+3y^2}{x^2+y^2} \leq \frac{3x^2+4y^2}{x^2+y^2} \leq \frac{4x^2+4y^2}{x^2+y^2}$, so $3\pi \leq \iint_R \frac{3x^2+4y^2}{x^2+y^2}\, dA \leq 4\pi$

Only $7\pi/2$ fits in this range, so the integral is $7\pi/2$.