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HW 1

1. $\sqrt{x-2} < x-4$

Because complex numbers are not comparable, we constrain $x \geq 2$. If $x < 4$, then RHS < 0 and the equation is not satisfiable

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Let $x \geq 4$ Then $x-4 \geq 0$, $x-2 \geq 2$

Also, since $f: x \mapsto \sqrt{x-2}$ is increasing in $[2, \infty)$,

$x \geq 4 \iff \sqrt{x-2} \geq \sqrt{2}$

Both $\sqrt{x-2}$ and $x-4$ are positive, and since $f: x \mapsto x^2$ is increasing in $[0 \infty)$,

$(\sqrt{x-2} < x-4) \iff (x \geq 4) \wedge (x-2 < x^2 - 8x + 16) \iff$

$(x \geq 4) \wedge (x^2 - 9x + 18 > 0) \iff (x \geq 4) \wedge (x-3)(x-6) \iff$

$(x \geq 4) \wedge (x < 3 \vee x > 6) \iff (x > 6)$

By double containment, $\{x \mid \sqrt{x-2} < x-4\} = (6, \infty)$

2. a) let $a \in \bigcap_{i=1}^{m} S_i$. This implies that

$\forall i \in [1, m]^{\mathbb{Z}}$, $a \in S_i$

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By the definition of an open set,

$\exists d_i \in (0, \infty)$ $N_g(a, d_i) \subseteq S_i$. Fix d_i

Consider $\min_{i \in [1,m]^{\mathbb{Z}}} (d_i) = d_{min}$. The minimum exists by finiteness.

By the definition of N_g, $\forall i \in [1, m]^{\mathbb{Z}}$ $N_g(a, d_{min}) \subseteq S_i$

Therefore, $N_g(a, d_{min}) \subseteq \bigcap_{i=1}^{m} S_i$

So this implies, $\bigcap_{i=1}^{m} S_i$ is an open set

b) The statement is false. Consider

$S_i = \{(x,y) \mid x^2 + y^2 < 1/i^2\}$

Each S_i is open because for all $a = (x,y)$ in S_i, the neighborhood $N_g(a, 1/i - \sqrt{x^2+y^2})$ is a subset of S_i

The intersection of $\bigcap_{i=1}^{\infty} S_i$ is the lone point $(0,0)$

because $\forall i \in \mathbb{N}$, $(0,0) \in S_i$ since $x^2+y^2 = 0^2+0^2 = 0 < 1/i^2$

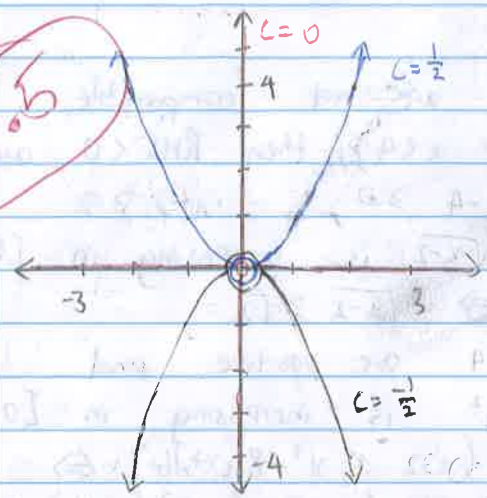
but $\forall a = (x,y) \neq (0,0)$, $a \notin S\lceil 1/\sqrt{x^2+y^2} \rceil$ since $\sqrt{x^2+y^2} \neq 0$

and $1/i^2 \leq (1/\,{}^1/\sqrt{x^2+y^2})^2 = x^2+y^2$

$\{(0,0)\}$ is not open since all neighborhoods contain points that are not in it. So though $\forall i \in \mathbb{N}$, S_i is open,

$\bigcap_{i=1}^{\infty} S_i$ is not open

3. +7.5



$x^2y/(x^4+y^2) = c.$ $\quad (x,y) \neq (0,0)$

$x^2y = c(x^4+y^2)$

$x^2y = cx^4 + cy^2$

$cy^2 - x^2y + cx^4 = 0$

$y = \dfrac{x^2 \pm \sqrt{x^4 - 4c^2x^4}}{2c}$ $\quad (c \neq 0)$

Requires $x^4 - 4c^2x^4 \geq 0$

$\Rightarrow x^4(1-4c^2) \geq 0$

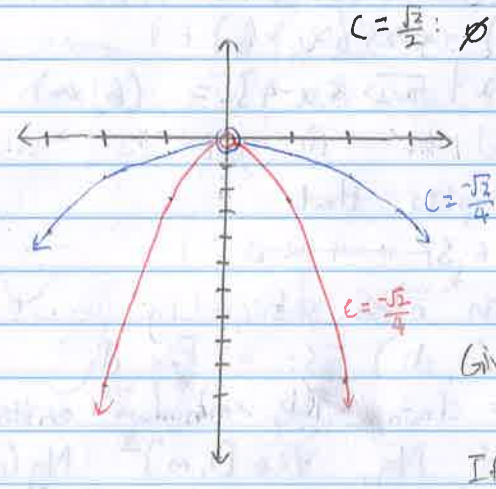
Note: $(1-4c^2) > 0 \Leftrightarrow 4c^2 < 1$

$\Leftrightarrow c \in (-\frac{1}{2}, \frac{1}{2})$

$c = \frac{\sqrt{2}}{2}: \emptyset$

$(1-4c^2) = 0 \Rightarrow c = \pm\frac{1}{2}$

$(1-4c^2) > 0 \Rightarrow c < -\frac{1}{2} \vee c > \frac{1}{2}.$



So if $c \in (-\frac{1}{2}, \frac{1}{2})$: $x \in (-\infty, \infty)$

$c = \pm\frac{1}{2}$: $x \in (-\infty, \infty)$

$c < -\frac{1}{2} \vee c > \frac{1}{2}$: $x = 0.$

$\Rightarrow y = 0 \Rightarrow \emptyset.$

Given that $\sqrt{\ldots}$ is defined,

$y = \left(\dfrac{1 \pm \sqrt{1-4c^2}}{2c}\right)x^2$ $\quad (c \neq 0)$

If $c = 0$: $x^2y = 0 \Rightarrow x = 0 \vee y = 0$

$\wedge ((x,y) \neq 0)$

$c = \frac{1}{2}$: $\sqrt{1-4c^2} = 0,\ y = x^2$ $\qquad c = \frac{-1}{2}$: $\sqrt{1-4c^2} = 0,\ y = -x^2$

$c = \frac{\sqrt{2}}{2}$: Impossible $\qquad c = \frac{-\sqrt{2}}{4}$: $\sqrt{1-4c^2} = \frac{\sqrt{2}}{2},$

$y = \left(\dfrac{\frac{\sqrt{2}}{2} \pm 1}{2(-\frac{\sqrt{2}}{4})}\right)x^2 = -(\sqrt{2} \pm 1)x^2$

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4. a.) $\lim_{v \to (0,0)} f(v) = 0 \qquad (x,y) \neq (0,0)$

Proof: Consider $\varepsilon > 0$. Let $v = (x,y)$

$x^2y^2 \leq (x^2+y^2)^2$ (add x^2 or y^2 respectively.)

So $3x^2y^2/(x^2+y^2) \leq 3(x^2+y^2)^2/(x^2+y^2) = 3(x^2+y^2)$

Let $\delta = \sqrt{\varepsilon/3}$ Consider $|v - \vec{0}| < \delta \Rightarrow \sqrt{x^2+y^2} < \delta \Rightarrow (x^2+y^2) < \delta^2$ by positivity

Then $f(v) \leq 3(x^2+y^2) < 3\delta^2 = 3\,\varepsilon/3 = \varepsilon$

so $f(v) < \varepsilon$. Note $|f(v) - 0| = f(v)$ by positivity

So the limit is 0.

b.) The limit doesn't exist at $(0,0)$

Let $C_1 = \{(x,y) \mid x = 2y\}$: $xy(x^2-y^2) = 2y^2(3y^2) = 6y^4$

$x^4 + y^4 = 17y^4$

Along C_1, $f(x,y) = 6/17 \qquad (x,y) \neq (0,0)$

C_1 contains the limit point $(0,0)$

Let $C_2 = \{(x,y) \mid x = 0\}$: $f(x,y) = 0$ along C_2

C_2 contains the limit point $(0,0)$.

The limit cannot exist because C_1 and C_2 intersect at $(0,0)$ but the values of f along the curves differ.

c.) $\lim_{v \to (0,0)} f(v) = 0. \qquad (x,y) \neq (0,0)$

Proof: $|x^3|y^4 = |y||x^3||y^3| = |y|\sqrt{x^6}\sqrt{y^6}$

$\leq |y|(\sqrt{x^6+y^6})^2 = |y|(x^6+y^6)$

so $|f(v)| = |x^3|y^4/(x^6+y^6) \leq |y| \leq \sqrt{y^2+x^2}$

Consider $\varepsilon > 0$. Let $\delta = \varepsilon$, consider $|v - \vec{0}| < \delta$

$\Rightarrow \sqrt{x^2+y^2} < \delta$

$|f(v)| \leq \sqrt{x^2+y^2} < \delta = \varepsilon \qquad$ so $|f(v) - \vec{0}| < \varepsilon$.

So the limit is 0.

d.) The limit doesn't exist at $(0,0)$

Let $C_1 = \{(x,y) \mid y = |x|^{2/3}\}$

$x^2y^3 = x^2x^2 = x^4, \quad x^4 + y^6 = x^4 + x^4 = 2x^4$

Along C_1, $f(v) = 1/2$. $\quad (x,y) \neq (0,0)$

C_1 contains the limit point $(0,0)$

Let $C_2 = \{(x,y) \mid x = 0\}$

Along C_2, $f(v) = 0$. $\quad (x,y) \neq (0,0)$

C_2 contains the limit point $(0,0)$

The limit cannot exist because C_1 and C_2 intersect at $(0,0)$ but the values of f, along the curves differ.