21-268 – Homework assignment week #10

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Reminder

Homework is due next Wednesdays before 5pm, to me in class or in Christopher Cox's mailbox in Wean Hall 6113 (pay attention to the arrow !). Late homework will never be accepted without a proper reason. In case of physical absence, electronic submissions by e-mail to both me your TA can be accepted. Please do not forget to write your name, andrew id and section and please use a staple if you have several sheets.

Exercises (27 pts)

1. (a) (1 pt) Let $C = \{(x, y, z) : z = x^2 + y^2, x = y\}$. Find $Y(x)$ and $Z(x)$ so that the vector function $\vec{r}(x) = x\vec{i} + Y(x)\vec{j} + Z(x)\vec{k}$ parametrizes C.

   (b) (1 pt)Let $C = \{(x, y, z) : z = x^2 + y^2, z = 4\}$. Find $Y(\theta)$ and $Z(\theta)$ so that the vector function $\vec{r}(\theta) = 2\sin(\theta)\vec{i} + Y(\theta)\vec{j} + Z(\theta)\vec{k}$ parametrizes C.

   (c) (2 pts)Let $S = \{(x, y, z) : 4x^2 + 4y^2 + z^2 = 4\}$. Find $Y(\phi, \theta)$ and $Z(\phi, \theta)$ so that the vector function $\vec{r}(\phi, \theta) = \sin(\phi)\cos(\theta)\vec{i} + Y(\phi, \theta)\vec{j} + Z(\phi, \theta)\vec{k}$ parametrizes S.

   (d) (2 pts) Let $S = \{(x, y, z) : x^2 + z^2 = 4\}$. Find $Y(\theta, y)$ and $Z(\theta, y)$ so that the vector function $\vec{r}(\theta, y) = 2\cos(\theta)\vec{i} + Y(\theta, y)\vec{j} + Z(\theta, y)\vec{k}$ parametrizes S. Think of cylindrical coordinates.

2. Let $f(r)$ be continuously differentiable and positive. Consider the surface, S, defined by

$$z = f(\sqrt{x^2 + y^2}) \text{ for } 0 \le a \le \sqrt{x^2 + y^2} \le b.$$

   (a) (1 pt) Define $\vec{R}(r, \theta) = r\cos(\theta)\vec{i} + r\sin(\theta)\vec{j} + f(r)\vec{k}$ for $a \le r \le b$ and $0 \le \theta \le 2\pi$. Show (in a few lines) that $\vec{R}$ describes the surface S defined above.

   (b) (3 pts) Compute the area of S. *Simplify the integral as much as you can without knowing f so that the answer is a single integral involving the derivative of f.*

3. (5 pts) Let $p > 0$ and $q > 0$ and consider $f(x, y) = x^{-p}y^{-q}$ on $R = \{(x, y) : 0 < x \le 1$ and $0 < y \le 1\}$. For what values of p and q is the improper integral

$$\iint_R f dA$$

   convergent? For what values of p and q is it divergent? Justify your answer.

4. (a) (2 pt) Suppose that $a(t, x)$, $b(t, x)$ and $f(t, x, y)$ all define $\mathcal{C}^1$ functions. What are

$$\frac{\partial}{\partial t}\int_{a(t,x)}^{b(t,x)} f(t, x, y)\mathrm{d}y \text{ and } \frac{\partial}{\partial x}\int_{a(t,x)}^{b(t,x)} f(t, x, y)\mathrm{d}y?$$

   This should be a very short generalization of the Leibniz rule seen in class.

(b) (4 pts) Assume that $g(t,x)$ defines a $\mathcal{C}^1$ function and let

$$u(t,x) = \int_{x-2t}^{x} g(t + (y-x)/2, y)\mathrm{d}y$$

Show that

$$u_t(t,x) + 2u_x(t,x) = 2g(t,x).$$

5. Let $f(x,y) = y$ and $C = \{(x,y) : 1 \leq x \leq 3 \text{ and } y = 1 + 2x\}$ oriented from $(1,3)$ toward $(3,7)$. Let $(x_0, y_0), \ldots, (x_n, y_n)$ be points on C with $1 = x_0 < x_1 < \cdots < x_n = 3$ and let $h = \max\left\{\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} : 1 \leq i \leq n\right\}$.

   (a) (2 pts) Let (x_i^*, y_i^*) be the points on C with $x_i^* = (x_i + x_{i-1})/2$ for $i = 1, 2, \ldots, n$ and show that

   $$\sum_{i=1}^{n} f(x_i^*, y_i^*)(x_i - x_{i-1}) = 10.$$

   (b) (2 pts) Now let (x_i^{**}, y_i^{**}) be any point on C between (x_{i-1}, y_{i-1}) and (x_i, y_i) for $i = 1, 2, \ldots, n$ and let

   $$R = \sum_{i=1}^{n} f(x_i^{**}, y_i^{**})(x_i - x_{i-1}).$$

   This is a Riemann sum for the integral $\int_C f\mathrm{d}x$. For any $\epsilon > 0$ find $\delta > 0$ such that $h < \delta$ implies $|R - 10| < \epsilon$.

6. (2 pts) Let $f(x,y) = xy$ and let C be the straight line segment from $(3,2)$ to $(1,-1)$. Compute

$$\int_C f\mathrm{d}s.$$