# 21-241 – Solution to Homework assignment week #7

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In this solution I just give you the reasoning. Of course, the computations of row reduction have to be done... but that's your job !

## Ex 1

1. $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$ reduces to $R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

   A basis for row $A$ is given by the non-zero row vectors of $R$, i.e.

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

   A basis for col $A$ is given by the columns of $A$ corresponding to leading columns of $R$, that is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

   $Ax = 0 \Leftrightarrow Rx = 0$ whose solutions are given by the free variable $x_3$ and the leading variables $x_4 = 0, x_2 = x_3, x_1 = -x_3$, i.e. the solutions are of the form $x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ so

$$\text{null } A = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

   Observe that, as we saw in class, the rank theorem is of course satisfied and row $A$ and col $A$ have the same dimension.

2. Now $A = \begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}$ reduces to $R = \begin{bmatrix} 1 & -2 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 3 & 7/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

   The same method yields

$$\text{row } A = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 7/2 \end{bmatrix} \right\} \qquad \text{col } A = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

   Solving $Rx = 0$, we get that $x_2, x_4, x_5$ are free and

$$x_1 = 2x_2 - x_4 - (1/2)x_5 \qquad x_3 = -3x_4 - (7/2)x_5$$

   so the solutions are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 - x_4 - (1/2)x_5 \\ x_2 \\ -3x_4 - (7/2)x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1/2 \\ 0 \\ -7/2 \\ 0 \\ 1 \end{bmatrix}$$

   and the three vectors above on the right-hand side form a basis of null $A$

# Ex 2

Among several possibilities, I chose to put these vectors as columns of a matrix $A$ and reduce it.

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

reduces to

$$R = \begin{bmatrix} 1 & 0 & 0 & 5/4 \\ 0 & 1 & 0 & 3/4 \\ 0 & 0 & 1 & 3/4 \end{bmatrix}$$

The leading columns are the three first ones, so I can extract a basis from the column space of $A$ by taking only the first three columns :

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

## Ex 3

1. No because we need 3 vectors to form a basis of $\mathbb{R}^3$.
2. We know that the dimensions of the row space and the columnspace agree and since $A$ has only 3 rows this dimension cannot be more than 3. But A has 5 columns so they must be linearly dependent (otherwise the dimension of the columnspace would be 5 !).
3. Just replace 5 by 4 in the result above and apply it to $A^T$ : its columns must be dependant, that is, the rows of $A$ must be dependent.
4. The rank of $A$ is $\leq 3$ so by the rank theorem, nullity $A$ can be $2, 3, 4$ or 5.
5. The rank of $A$ is $\leq 2$ so by the rank theorem, nullity $A$ can be $0, 1$ or 2.

## Ex 4

We have a set of 4 vectors in $\mathbb{R}^4$ : being a basis is equivalent to spanning $\mathbb{R}^4$ and to being independent. It is also equivalent to saying that the matrix with these vectors as columns is invertible. Compute the reduced row echelon form and observe that it is $I_n$. Thus the columns form a basis of $\mathbb{R}^4$.