21-241 – Solution to Homework assignment week #5

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Ex 1

1. Let us prove that for all $n \times n$ matrices A, $(A^T)^T = A$.

   Fix $1 \leq i, j \leq n$. Then by definition of transpose $((A^T)^T)_{ij} = (A^T)_{ji} = A_{ij}$. This is true for all $1 \leq i, j \leq n$ so $(A^T)^T = A$.

2. Similarly, $(\lambda A)^T_{ij} = (\lambda A)_{ji} = \lambda A_{ji} = \lambda (A^T)_{ij}$

3. Similarly, $(A+B)^T_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = A^T_{ij} + B^T_{ij} = (A^T + B^T)_{ij}$.

Ex 2

1. We obtain $B^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B^3 = 0$.

2. We have $A = I_3 + B$.

3. $A^n = (I_3 + B)^n$. Now, **since** I_3 **and** B **commute**, that is, $I_3 B = B I_3$, we are allowed to use Newton's binomial formula to get for all $n \geq 2$

$$A^n = \sum_{k=0}^{n} \binom{n}{k} B^k I_3^{n-k} = \sum_{k=0}^{n} \binom{n}{k} B^k = \binom{n}{0} B^0 + \binom{n}{1} B^1 + \binom{n}{2} B^2$$

   since all the next terms are 0 by the above. This yields

$$A^n = I_3 + nB + \frac{n(n-1)}{2} B^2 = \begin{bmatrix} 1 & n & n(n+1)/2 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$$