1) $\bullet$ $W = \left\{ \begin{bmatrix} t/2 \\ -t/2 \\ 2t \end{bmatrix} \middle/ t \in \mathbb{R} \right\} = \operatorname{span}\left( \begin{bmatrix} 1/2 \\ -1/2 \\ 2 \end{bmatrix} \right) = \operatorname{Span} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$.

Let $A = [1 \ -1 \ 4]$. $W = \operatorname{row}(A)$.

So $W^\perp = \operatorname{null}(A)$.

$[1 \ -1 \ 4 \ | \ 0] \leftarrow$ solutions are $\begin{pmatrix} y-4z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$

So $\boxed{W^\perp = \operatorname{span}\left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right)}$

$\bullet$ $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle/ 2x - y + 3z = 0 \right\}$. W is defined by $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

so $\boxed{W^\perp = \operatorname{span}\left( \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right)}$.

$\bullet$ $W^\perp = \operatorname{null}\left( \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix} \right)$.

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & 0 \\ 0 & 1 & -2 & 1 & 0 \end{array}\right] \xleftarrow[R_1 \leftarrow R_1 + R_2]{} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 3 & 0 \\ 0 & \boxed{1} & -2 & 1 & 0 \end{array}\right]$$

so solutions are $\begin{array}{l} x = -z - 3t \\ y = 2z - t \end{array} \rightsquigarrow \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = z\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

so $\boxed{W^\perp = \operatorname{span}\left( \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right)}$.

2) Yes. First, $v_{k+1}, \dots, v_n \in W^\perp$ because they are orthogonal to each $v_1, \dots, v_k$ so to $W = \operatorname{span}(v_1, \dots, v_k)$.

Moreover $(v_{k+1}, \dots, v_n)$ is orthogonal and $\neq 0$ and contains $n-k$ vectors, i.e $\dim(W^\perp)$ vectors. It is then an orth. basis for it.

3) (a)+(b). We know that $x$ decomposes in a unique way

$$x = \underbrace{\operatorname{proj}_W(x)}_{\in W} + \underbrace{(x - \operatorname{proj}_W(x))}_{\in W^\perp}$$

~~The only way to have $x \in W$~~

Knowing that $W \cap W^\perp = \{0\}$ we have that

$$x \in W \iff \begin{cases} \mathrm{proj}_W(x) = x \\ x - \mathrm{proj}_W(x) = 0 \end{cases} \iff x = \mathrm{proj}_W(x).$$

Similarly $x \in W^\perp \iff x - \mathrm{proj}_W(x) = x \iff \mathrm{proj}_W(x) = 0.$

(c) $\mathrm{proj}_W(x) \in W$ for all $x \in \mathbb{R}^N$, so

$$\mathrm{proj}_W(\underbrace{\mathrm{proj}_W(x)}_{\in W}) = \mathrm{proj}_W(x).$$

3) Pick $v_1 = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$. A smart choice for $v_2$ is any vector orth. to $v_1$, i.e. $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying $3x + y + 5z = 0$, so $v_2 = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$ for instance.

Then I can add $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. $\begin{bmatrix} 3 & 1 & 0 \\ 1 & -3 & 0 \\ 5 & 0 & 1 \end{bmatrix}$ is invertible (det. $-10$) and $v_1 \perp v_2$ already so I just need to replace $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ by

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{5}{35}\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} - \cancel{\frac{0}{10}\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -3/7 \\ -1/7 \\ 2/7 \end{bmatrix} \quad \text{so}$$

$$\left( \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} \right) \text{ works out.}$$