21-241 Matrices and Linear Transformations Lecture 3
Midterm #1

October 3, 2016

Name ______________________________

Andrew ID ______________________________

Section ______________________________

Time: 50 minutes

This midterm should contain 4 exercises. No textbook, calculator, recitation or exterior material is authorized. You can use your lecture notes but be aware that each time you go looking into them, you lose time. All statements should be proved rigorously. You have of course the right to use the results (theorems, remarks, examples...) provided in class. All the exercises are independent.

Question:	1	2	3	4	Total
Points:	12	12	8	18	50
Score:					

1. (a) (8 points) Given $a, b, c \in \mathbb{R}$, solve

$$(S)\begin{cases} x + \ \ y + z = a \\ x - \ \ y + z = b \\ \quad\ 2y - z = c \end{cases}$$

by Gauß-Jordan elimination (that is, using reduced row echelon form).

(b) (4 points) Show that

$$\mathbb{R}^3 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

2. Let $(\mathcal{P}_1)$ resp. $(\mathcal{P}_2)$ be planes in $\mathbb{R}^3$ defined by equations in normal forms

$$n_1 \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1 \qquad n_2 \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -2$$

where $n_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $n_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) (4 points) Without doing any computation, discuss what the intersection $(\mathcal{P}_1) \cap (\mathcal{P}_2)$ is geometrically (justify your answer).

(b) (8 points) Compute this intersection and write it as an affine space.

3. (8 points) Find all possible linear combinations of $\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -1 \\ -3 \end{bmatrix}$ that are equal to $\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$.

4. We define the *trace* of a square $n \times n$ matrix to be the sum of the diagonal terms :

$$\mathrm{tr} : \mathcal{M}_{nn}(\mathbb{R}) \to \mathbb{R} \qquad A \mapsto \sum_{i=1}^{n} a_{ii}$$

   (a) (5 points) Prove that tr is a linear transformation.

   (b) (2 points) Let (α_{ij}) denote a family of real numbers indexed by $1 \leq i, j \leq n$. Explain briefly why

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{n} \alpha_{ij} \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \alpha_{ij} \right).$$

   (c) (6 points) Prove that for all $A, B \in \mathcal{M}_{nn}(\mathbb{R})$, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
   Hint: use the above identity.

   (d) (5 points) Find examples of matrices such that $\mathrm{tr}(ABC) \neq \mathrm{tr}(ACB)$.

Remark. One can prove that *tr* is invariant under cyclic permutations, that is

$$\mathrm{tr}(A_1 A_2 \cdots A_n) = \mathrm{tr}(A_k A_{k+1} \cdots A_n A_1 \cdots A_{k-1})$$

for all $1 \leq k \leq n$. Observe that ACB is not a cyclic permutation of ABC.