Lecture Notes
for
21-260 Differential Equations

D.G.C. Handron

January 12, 2007

Chapter 1

Introduction

1.1 Lecture 1. Introduction to Differential Equations

1.1.1 What is a differential equation?

Simply put, a differential equation is an equation involving the derivative of a function. For example

$$\frac{dy}{dt} = 2y \tag{1.1.1}$$

and

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = 0 \tag{1.1.2}$$

are examples of differential equations.

In each case y represents a function of the variable t. Since the value of y depends on the value of t, y is called the *dependent variable*, even though it is a function. The variable t is called the *independent variable*.

1.1.2 What is a solution to a differential equation?

A solution to a differential equation is a function that, when substituted into the equation, produces a true statement. For example, if $y_1(t) = e^{2t}$, then

$$\frac{dy_1}{dt}(t) = 2e^{2t} \tag{1.1.3}$$

and

$$2y_1(t) = 2e^{2t}. \tag{1.1.4}$$

Thus, y_1 is a solution to the differential equation 1.1.1, because

$$\frac{dy_1}{dt} = 2y_1. \tag{1.1.5}$$

Similarly, if $y_2(t) = e^t \cos(t)$, then

$$\frac{dy_2}{dt}(t) = e^t(\cos(t) - \sin(t)), \tag{1.1.6}$$

and

$$\frac{d^2y_2}{dt^2}(t) = e^t(-2\sin(t)). \tag{1.1.7}$$

So

$$\begin{aligned} &\tfrac{d^2y_2}{dt^2} - 2\tfrac{dy_2}{dt} + 2y \\ &= e^t(-2\sin(t)) - 2e^t(\cos(t) - \sin(t)) + 2e^t\cos(t) \\ &= 0. \end{aligned} \tag{1.1.8}$$

We see, then, that y_2 is a solution to equation 1.1.2.

1.1.3 What types of differential equations are there?

Because any particular technique used in the study of differential equations tends to apply only in restricted cases, there are many ways of categorizing differential equations.

Order

The *order* of a differential equation is the degree of the highest derivative found in the equation. Of the examples listed above, 1.1.1 is a *first order* equation, while 1.1.2 is a *second order* equation.

In general an n-th order differential equation is an expression of the form

$$F(t, y, \frac{dy}{dt}, \ldots, \frac{d^ny}{dt^n}) = 0. \tag{1.1.9}$$

For the examples above, the expressions are

$$F_1(t, y, \frac{dy}{dt}) = \frac{dy}{dt} - 2y, \tag{1.1.10}$$

and

$$F_2(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}) = \frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y \tag{1.1.11}$$

respectively.

Linear vs. non-linear

A differential equation

$$(t, y, \frac{dy}{dt}, \ldots, \frac{d^ny}{dt^n}) = 0. \tag{1.1.12}$$

is said to be *linear* if F is a linear function of the dependent variable y and all of its derivatives. The equations 1.1.1 and 1.1.2 are both linear for example.

Note that a linear differential equation need not be linear in the independent variable.

$$t^2 \frac{d^2y}{dt^2} - e^{t^2} \frac{dy}{dt} + \sin(e^{t^2})y = 0 \tag{1.1.13}$$

is a linear equation. On the other hand

$$\frac{dy}{dt} + 2\sin(y) = 0 \tag{1.1.14}$$

is not a linear differential equation, since $2\sin(y)$ is not a linear function of y. Similarly

$$y\frac{dy}{dt} = 0 \tag{1.1.15}$$

is not a linear differential equation, since $y\frac{dy}{dt}$ is not a linear expression.

Ordinary Differential Equations and Partial Differential Equations

If a quantity depends on two or more variables that are each independent, its behavior can be described by an equation involving partial derivatives. The result is then a *partial differential equation* (PDE). For example the temperature along a thin rod may depend on both the position along the rod and the time that has elapsed since the beginning of the experiment. In fact

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1.1.16}$$

describes just such a situation. Here $u(x,t)$ represents the temperature at a position x along the rod at time t. The parameter α describes how easily heat energy travels along the rod (i.e. its thermal conductivity).

Differential equations that involve only ordinary derivatives (rather than partial derivatives) are referred to as *ordinary differential equations* (ODE's).

1.1.4 Systems of differential equations

All of the examples we have considered involve a single dependent variable. It is possible for there to be several quantities that all depend on a single independent variable, and possibly on each other, too. The pair of differential equations

$$\begin{array}{rcl} \frac{dx}{dt} & = & y \\ \frac{dy}{dt} & = & -x \end{array} \tag{1.1.17}$$

Must be treated in unison. It is not possible to solve for x without some knowledge of the behavior of y, but y cannot be determined without knowing something about x.

The pair of equations 1.1.17 is called a *system of differential equations*. Systems of differential equations frequently arise in modeling problems. For instance the system

$$\begin{aligned} \tfrac{dR}{dt} &= \alpha R - \beta RF \\ \tfrac{dF}{dt} &= -\gamma F + \delta RF \end{aligned} \tag{1.1.18}$$

represents the interaction between two animal populations, a predator F (foxes?) and a prey R (rabbits?).

1.1.5 Initial Value Problems

As a general rule, a differential equation

$$\frac{dy}{dx} = f(x, y) \tag{1.1.19}$$

will have many different solutions. In many instances, we are not interested in all the possible solutions. We only wish to consider solutions that meet some particular conditions. Frequently, we are interested in a solution that satisfies a given initial condition. An *initial condition* $y(x_0) = y_0$ determines the value y_0 the dependent variable must take for the specified value x_0 of the dependent variable.

Chapter 2

First Order Differential Equations

2.1 Lecture 2. Analytic Technique: Separable Equations

2.1.1 Definition

We say that the differential equation

$$\frac{dy}{dx} = f(x, y) \tag{2.1.1}$$

is a *separable equation* if the the function f can be factored as $f(x, y) = g(x)h(y)$.

2.1.2 Integration

Taking a cue from elementary algebra, we may try to work toward a solution by moving all the y's to the left hand side of the equation, to get

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x). \tag{2.1.2}$$

There are some problems with this if $h(y)$ is ever zero, but for now we'll ignore this. Remembering that the dependent variable y is actually a function $y(x)$, we see that both sides of equation 2.1.2 are actually functions of x. We can integrate both sides with respect to x:

$$\int \frac{1}{h(y(x))}\frac{dy}{dx}dx = \int g(x)dx. \tag{2.1.3}$$

The integral on the left can be simplified with the u-substitution $u = y(x)$. Then $du = \frac{dy}{dx}dx$, and

$$\int \frac{1}{h(u)}du = \int g(x)dx. \tag{2.1.4}$$

Of course, the choice of the letter u doesn't have any real significance. We can choose any letter we like, y for instance, to get

$$\int \frac{1}{h(y)}dy = \int g(x)dx. \tag{2.1.5}$$

If we can compute both of these integrals, we are in good shape.

Now, it may seem that this u-substitution is a big waste of time. Why not just cancel the dx's? Well, in practice, that is fine. This is the reason that the $\frac{dy}{dx}$ notation has become popular. But it should be kept in mind that $\frac{dy}{dx}$ is not really a fraction, and dx is not really a number.

Example 1. *Find a solution to the differential equation*

$$\frac{dy}{dx} = x(y-3). \tag{2.1.6}$$

Since the right hand side can be factored with $g(x) = x$ and $h(y) = y-3$, we can rewrite it as

$$\frac{1}{y-3}\frac{dy}{dx} = x \tag{2.1.7}$$

then integrate

$$\int \frac{1}{y-3}\frac{dy}{dx}dx = \int \frac{1}{y-3}dy = \int xdx \tag{2.1.8}$$

Integrating, we see

$$\begin{aligned} \ln|y-3| &= \tfrac{1}{2}x^2 + C \\ |y-3| &= e^{\frac{1}{2}x^2+C} = e^C e^{\frac{1}{2}x^2} \\ y-3 &= \pm e^C e^{\frac{1}{2}x^2} \end{aligned} \tag{2.1.9}$$

Since e^C is always a positive constant, we can solve for y to get

$$y(x) = ke^{\frac{1}{2}x^2} + 3, \tag{2.1.10}$$

where k is a non-zero constant. So we see that we have a whole family of solutions.

Example 2. *Find solutions to the initial value problem*

$$\frac{dy}{dx} = x(y-3), \quad y(0) = 7. \tag{2.1.11}$$

To solve for the initial condition, note that we require

$$y(0) = ke^{\frac{1}{2}(0)^2} + 3 = k(1) + 3 = 7 \tag{2.1.12}$$

and so we must take $k = 4$. *The solution to this initial value problem is*

$$y(x) = 4e^{\frac{1}{2}x^2} + 3, \tag{2.1.13}$$

Example 3. *Find solutions to the initial value problem*

$$\frac{dy}{dx} = x(y-3), \quad y(2) = 3. \tag{2.1.14}$$

To solve for the initial condition, note that we require

$$y(2) = ke^{\frac{1}{2}(2)^2} + 3 = ke^2 + 3 = 3 \tag{2.1.15}$$

and so we must take $k = 0$. *This is troubling, since* k *was to be a non-zero constant. But, ignoring that for the moment, we get*

$$y(x) = (0)e^{\frac{1}{2}x^2} + 3 = 3. \tag{2.1.16}$$

There are two important things to note here. First, if $y = 3$, *then* $h(y) = 0$, *and we could not have divided by* $h(y)$ *in the first place. That having been said, notice that*

$$\frac{d}{dx}[3] = 0 \tag{2.1.17}$$

and

$$x(3-3) = 0 \tag{2.1.18}$$

so $y(x) = 3$ *is, in fact, a solution.*

When dealing with a separable equation

$$\frac{dy}{dx} = g(x)h(y), \tag{2.1.19}$$

any value y_0 for which $h(y_0) = 0$ corresponds to a constant solution $y(x) = y_0$. These solutions may not correspond to any choice of the arbitrary constant that appears in the integration.

Example 4. *Solve the initial value problem*

$$\frac{dy}{dx} = -2xy^2, \quad y(0) = 0. \tag{2.1.20}$$

Following the procedure above

$$\begin{aligned} \frac{1}{y^2}\frac{dy}{dx} &= -2x \\ \int \frac{1}{y^2}dy &= \int -2x dx \\ -\frac{1}{y} &= -x^2 + C \\ y(x) &= \frac{1}{x^2+C} \end{aligned} \tag{2.1.21}$$

Note that in the last line the constant C seems to have changed sign. It's an arbitrary constant, though, so this doesn't really matter. Throughout these notes, we'll treat arbitrary constants with a similarly cavalier attitude.

Solving for the initial condition, we try

$$y(0) = \frac{1}{(0)^2 + C} = \frac{1}{C} = 0. \tag{2.1.22}$$

Unfortunately, no value of C will work. Looking back, we see that for $y = 0$, the right hand side of 2.1.20 becomes

$$-2x(0)^2 = 0. \tag{2.1.23}$$

In fact, $y(x) = 0$ is a constant solution to 2.1.20.

2.1.3 Implicit Solutions

Unfortunately, it is not always possible, after integrating, to solve the resulting expression for y.

Example 5. *Solve the differential equation*

$$\frac{dy}{dx} = \frac{3x^2 + 2}{5y^4 + 4y^3 + 2y}. \tag{2.1.24}$$

The equation is separable, with $h(y) = \frac{1}{5y^4+4y^3+2y}$. So

$$\begin{aligned} \int (5y^4 + 4y^3 + 2y)dy &= \int (3x^2 + 2)dx \\ y^5 + y^4 + y^2 &= x^3 + 2x + C \end{aligned} \tag{2.1.25}$$

Since there is no general solution for a fifth degree polynomial, there is no way to get an explicit expression for $y(x)$.

The relationship

$$y^5 + y^4 + y^2 = x^3 + 2x + C \tag{2.1.26}$$

implicitly describes a relationship between x and y. This is referred to as an *implicit solution.*

To summarize: In order to solve the separable equation

$$\frac{dy}{dx} = g(x)h(y), \tag{2.1.27}$$

1. divide by $h(y)$ and integrate to get a family of solutions (which may or may not include all constant solutions).
2. Each value y_0 such that $h(y_0) = 0$ represents a constant solution

$$y(x) = y_0. \tag{2.1.28}$$

It should be remembered that

- You may not be able to perform the integrations.
- You may not be able to solve for y explicitly in terms of x.

2.1.4 General Solutions and Particular Solutions

Definition 1. A *general solution* to a differential equation

$$\frac{dy}{dx} = f(x, y) \tag{2.1.29}$$

is a family of solutions that provides a solution for every initial value problem.

For separable equations, the general solution consists of the solutions found in 1 above, together with the constant solutions from 2.

Example 6. *The general solution to the differential equation* $\frac{dy}{dx} = -2xy^2$ *is*

$$\begin{array}{c} y(x) = \frac{1}{x^2+C} \\ or y(x) = 0. \end{array} \tag{2.1.30}$$

Definition 2. A *particular solution* is a single function that satisfies an initial value problem. i.e. a single member of the family in the general solution.

2.1.5 Mixing Problems

Example 7. *In this example we will investigate a polluted pond. The assumptions we will make are:*

- *The pond holds 1000* m^3 *of water.*
- *The pond is initially (at* $t = 0$*) unpolluted.*
- *A stream containing mercury (Hg) flows into the pond. The rate of flow of the stream is 3* m^3*/hour. The concentration of mercury in the stream is* $e^{-t/100}$ g/m^3*.*
- *Another stream flows out of the pond, at a rate of 3* m^3*/hour.*

- *Currents within the pond keep the water well mixed, ensuring that the concentration of mercury is uniform throughout the pond.*

We can let $H(t)$ denote the amount (in grams) of mercury at time t (measured in hours). There is only one way mercury can enter the pond: from the polluted stream. Similarly, it can leave only from the other stream.

The rate at which mercury is added is

$$[\textit{flow rate}] \times [\textit{concentration}] = [3\ m^3/\textit{hour}] \times [e^{-t/100}\ g/m^3] = 3e^{-t/100} g/\textit{hour} \tag{2.1.31}$$

This gives a rate of $3e^{-t/100}$*g/hour.*

The rate at which mercury is removed from the pond is

$$[\textit{flow rate}] \times [\textit{concentration}] = [3\ m^3/\textit{hour}] \times \frac{[H(t)\ g]}{[1000\ m^3]} \tag{2.1.32}$$

This gives a rate of $\frac{3H(t)}{1000}$ *g/hour.*

Combining these, we see

$$\frac{dH}{dt} = 3e^{-t/100} - \frac{H(t)}{1000}, \tag{2.1.33}$$

where the units on both sides are g/hour. This is a linear equation. It can be rewritten as

$$\frac{dH}{dt} + \frac{3}{1000}H = 3e^{-t/100}. \tag{2.1.34}$$

With $p(t) = \frac{3}{1000}$, *we compute*

$$\mu(t) = e^{\int \frac{3}{1000} dt} = e^{\frac{3}{1000}t}. \tag{2.1.35}$$

So

$$\frac{d}{dt}\left(e^{\frac{3}{1000}t} y\right) = 3e^{-t/100} e^{\frac{3}{1000}t} = 3e^{-\frac{7}{1000}t}. \tag{2.1.36}$$

Integrating

$$e^{\frac{3}{1000}t} y = -\frac{3000}{7} e^{-\frac{7}{1000}t} + C. \tag{2.1.37}$$

Multiplying thorough by $\mu(t)$ *finally yields*

$$y(t) = Ce^{-\frac{3}{1000}t} - \frac{3000}{7} e^{-t/100} \tag{2.1.38}$$

Now, solving for the initial condition $H(0) = 0$, *we get* $C = \frac{3000}{7}$, *and*

$$y(t) = \frac{3000}{7}\left(e^{-\frac{3}{1000}t} - e^{-t/100}\right) \tag{2.1.39}$$

2.2 Lecture 3. Qualitative Method: Direction Fields, and a Few Mathematical Models.

2.2.1 Direction Fields

It is possible to sketch curves without computing solutions. Suppose we wish to sketch solutions for the differential equation

$$\frac{dy}{dt} = f(t, y) \tag{2.2.1}$$

If some function $y(t)$ is a solution to 2.2.1, then

$$\frac{dy}{dt}(t) \tag{2.2.2}$$

is the slope of the solution curve $y = y(t)$

Since y is a solution to 2.2.1 it must satisfy

$$\frac{dy}{dt}(t) = f(t, y(t)). \tag{2.2.3}$$

So the slope of a solution curve passing through the point t, y in the ty-plane is completely determined by the values of t and y. The value $f(t_0, y_0)$ is the slope of any solution passing through the point (t_0, y_0).

We can use this fact to get an idea how solutions behave without computing them analytically.

Example 8. *We can get an idea how solutions to the differential equation*

$$\frac{dy}{dx} = \frac{1}{2}y \tag{2.2.4}$$

by choosing an array of points in the ty-plane and making a short line segment for each point. The line segment at (t_0, y_0) will have slope $f(t_0, y_0)$. The results of this are shown in Figure 8

Using the direction field, it is possible to sketch solutions, by simply following the direction field lines, moving parallel to the nearest lines as shown in Figure 8.

Example 9. *In this example, we will see how qualitative and analytic techniques may be used together to gain a greater understanding of the behavior of solutions. Consider the direction field for $y' = t+2y$, as shown in Figure 9.*

Some of the solutions are increasing as they leave the rectangle, others are decreasing. We might like to find a solution curve that separates the solutions that eventually increase from those that eventually decrease.

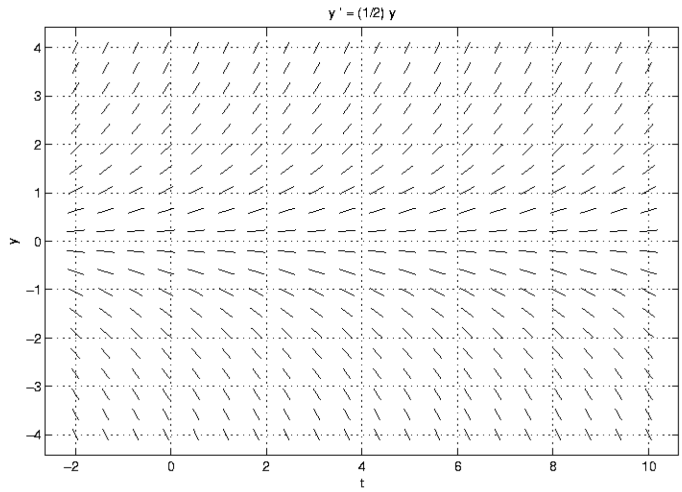

Figure 2.1: Direction field for $y' = \frac{1}{2}y$.

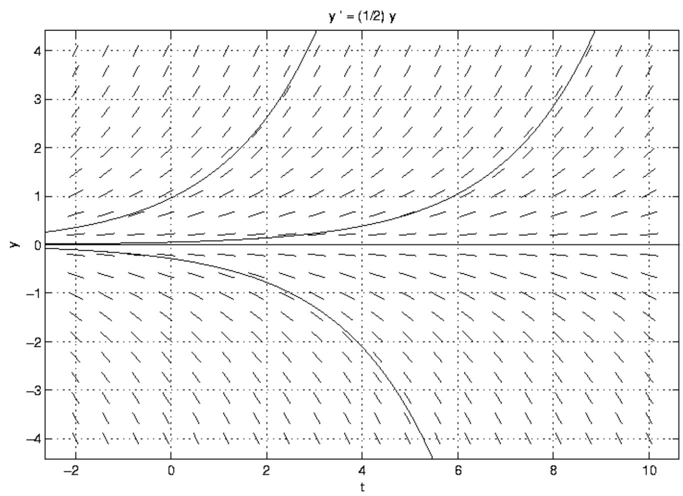

Figure 2.2: Direction field and some solution curves for $y' = \frac{1}{2}y$.

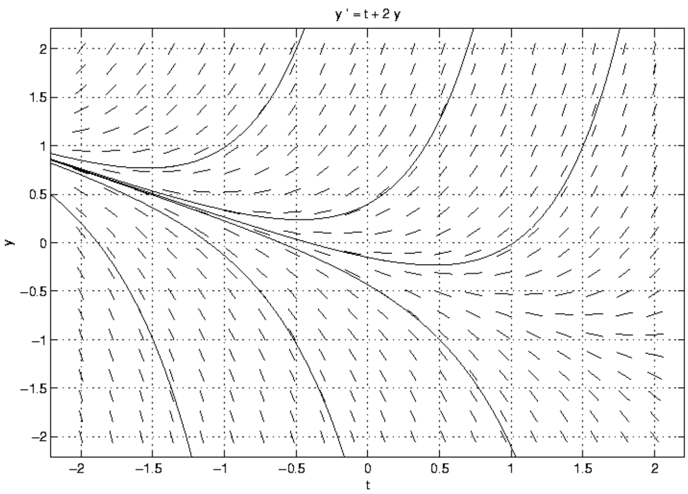

Figure 2.3: Direction field and some solution curves for $y' = t + 2y$.

First note that the direction field lines are constant along lines with slope $-\frac{1}{2}$. *Why is this so? Recall that the slope of solution curves, and hence the direction field lines, is given by*

$$f(t, y) = t + 2y. \tag{2.2.5}$$

Where will the direction field lines have a given slope, say m*? Wherever*

$$t + 2y = m, \tag{2.2.6}$$

i.e. along the line

$$y = -\frac{1}{2}t + \frac{1}{2}m \tag{2.2.7}$$

If we can find a line in the plane such that the direction field along that line is parallel to that line, we might expect a solution curve to follow that line. We can find such a line by setting $m = -\frac{1}{2}$ *to get*

$$y(t) = -\frac{1}{2}t - \frac{1}{4} \tag{2.2.8}$$

We can verify that this line is a solution by checking

$$\frac{dy}{dt} = \frac{d}{dt}\left[-\frac{1}{2}t - \frac{1}{4}\right] = -\frac{1}{2}, \tag{2.2.9}$$

and

$$f(t, y(t)) = t + 2y(t) = t + 2\left(-\frac{1}{2}t - \frac{1}{4}\right) = t - t - \frac{1}{2} = -\frac{1}{2}. \tag{2.2.10}$$

Figure 9 shows this straight line solution separating the decreasing solutions from those that eventually increase.

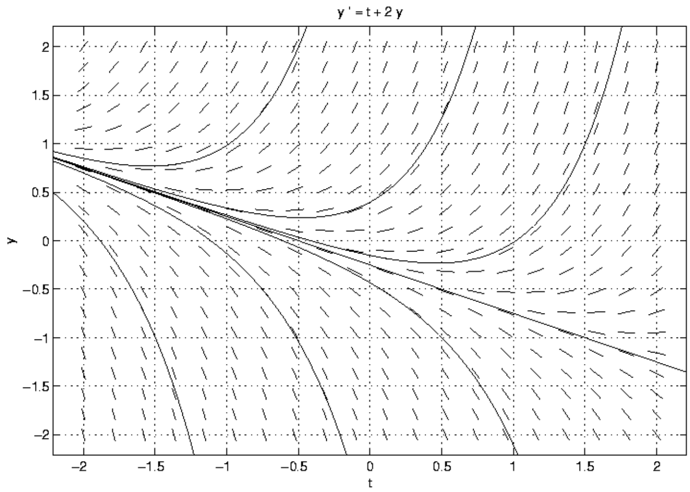

Figure 2.4: A straight line solution separates (eventually) increasing and decreasing soltuions of $y' = t + 2y$.

2.2.2 Newton's Law of Cooling

2.2.3 Toricceli's Law

2.3 Lecture 4. Analytic Method: First Order Linear Equations

The differential equation

$$\frac{dy}{dt} + p(t)y = g(t) \tag{2.3.1}$$

is linear in both y and $\frac{dy}{dt}$. It is a first order linear differential equations. We would like to be able to find an explicit solution for differential equations of this form. We shall do this by putting the equation into a form that allows us to integrate.

2.3.1 Integrating Factors

First consider the left hand side (LHS):

$$\frac{dy}{dt} + p(t)y. \tag{2.3.2}$$

After staring at this for a sufficiently long time, with sufficient motivation, you might start to think that it looks a little like the result of using the product rule – there are two term, one involves y and the other involves $\frac{dy}{dt}$.

Lets think about what we get when we apply the product rule to a product:

$$\frac{d}{dt}\left(\mu(t)y(t)\right) = \mu(t)\frac{dy}{dt} + \left(\frac{d\mu}{dt}\right)y. \tag{2.3.3}$$

Comparing

$$\mu(t)\frac{dy}{dt} + \left(\frac{d\mu}{dt}\right)y \tag{2.3.4}$$

to the LHS of our equation

$$\frac{dy}{dt} + p(t)y, \tag{2.3.5}$$

we see that the $\frac{dy}{dt}$ terms have different coefficients. We can remedy this by multiplying the LHS by $\mu(t)$ (whatever that might be) to get

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y. \tag{2.3.6}$$

This will make the two expressions identical, provided we can chose a $\mu(t)$ such that

$$\frac{d\mu}{dt} = \mu(t)p(t). \tag{2.3.7}$$

This is a separable equation, and so we can solve it using the techniques in Section 2.1. Separating and integrating

$$\int \frac{\mu}{d}\mu = \int p(t)dt \tag{2.3.8}$$

we get

$$\mu(t) = e^{\int p(t)dt}. \tag{2.3.9}$$

2.3.2 General Solutions

So multiplying the linear differential equation by $\mu(t)$ allows us to rewrite the equation as

$$\begin{aligned} \mu(t)\tfrac{dy}{dt} + \mu(t)p(t)y &= \mu(t)g(t) \\ \tfrac{d}{dt}\left(\mu(t)y(t)\right) &= \mu(t)g(t). \end{aligned} \tag{2.3.10}$$

Integrating

$$\int \frac{d}{dt}\left(\mu(t)y(t)\right)dt = \int \mu(t)g(t)dt \tag{2.3.11}$$

we get

$$\mu(t)y(t) = \int \mu(t)g(t)dt \tag{2.3.12}$$

If the integral on the right can be solved, then dividing by $\mu(t)$ produces an explicit expression for $y(t)$.

Example 10. *Solve the initial value problem*

$$\begin{aligned} \tfrac{dy}{dt} + ty &= t \\ y(t_0) &= y_0 \end{aligned} \tag{2.3.13}$$

This is a linear first order differential equation with

$$p(t) = t, \qquad \text{and } g(t) = t, \tag{2.3.14}$$

so

$$\mu(t) = e^{\int p(t)dt} = e^{\int t dt} = e^{\frac{1}{2}t^2 + C} \tag{2.3.15}$$

Since we need only one such function, we can take (for convenience) $C = 0$, and use $\mu(t) = e^{\frac{1}{2}t^2}$.

Multiplying by $\mu(t)$ we get

$$e^{\frac{1}{2}t^2}\frac{dy}{dt} + te^{\frac{1}{2}t^2}y = te^{\frac{1}{2}t^2}, \tag{2.3.16}$$

or, because of the way we chose μ

$$\frac{d}{dt}\left(\mu(t)y(t)\right) = te^{\frac{1}{2}t^2}. \tag{2.3.17}$$

Integrating, we get

$$\mu(t)y(t) = \int te^{\frac{1}{2}t^2}dt = e^{\frac{1}{2}t^2} + C. \tag{2.3.18}$$

Solving for y, we find

$$y(t) = 1 + Ce^{-\frac{1}{2}t^2} \tag{2.3.19}$$

We can use this to solve for the initial condition $y(t_0) = y_0$:

$$y(t_0) = 1 + Ce^{-\frac{1}{2}t_0^2} = y_0. \tag{2.3.20}$$

Solving for C gives

$$C = (y_0 - 1)e^{\frac{1}{2}t_0^2}, \tag{2.3.21}$$

and

$$\begin{aligned} y(t) &= 1 + (y_0 - 1)e^{\frac{1}{2}t_0^2}e^{-\frac{1}{2}t^2} \\ &= 1 + (y_0 - 1)e^{\frac{1}{2}(t_0^2 - t^2)} \end{aligned} \tag{2.3.22}$$

Since the solution $y(t) = 1 + Ce^{-\frac{1}{2}t^2}$ found in Example 10 can be used to solve any initial value problem, it is, in fact, the general solution for the differential equation.

In fact, this method, when it works, will always lead to a general solution for the differential equation. There is no need to worry about "disappearing solutions" as there is in the case of separable equations.

Not every linear first order equation is of the form

$$y' + p(t)y = g(t). \tag{2.3.23}$$

You may, from time to time encounter equations of the form

$$P(t)y' + Q(t)y = R(t). \tag{2.3.24}$$

These can be handled by simply dividing though by $P(t)$, to get

$$y' + \frac{Q(t)}{P(t)}y = \frac{R(t)}{P(t)}. \tag{2.3.25}$$

This is of the desired form with

$$p(t) = \frac{Q(t)}{P(t)} \tag{2.3.26}$$

and

$$g(t) = \frac{R(t)}{P(t)}. \tag{2.3.27}$$

Example 11. *Find the general solution to*

$$ty' - y = t^2 e^{-t}. \tag{2.3.28}$$

Since the coefficient of y' *is* $P(t) = t$*, we must divide through to get*

$$y' - \frac{1}{t}y = te^{-t}, \tag{2.3.29}$$

which is of the desired from, with $p(t) = -\frac{1}{t}$ *and* $g(t) = te^{-t}$*.*

Since

$$e^{\int -\frac{1}{t}dt} = e^{-\ln|t|} = e^{\ln|\frac{1}{t}|} = |\frac{1}{t}| \tag{2.3.30}$$

(we've assumed the constant of integration is $C = 0$*.) We may want to try*

$$\mu(t) = \frac{1}{t} \tag{2.3.31}$$

This is not exactly what we computed above (no absolute value) but it must work when $t > 0$*. Maybe it will work for all* t*. Let's check:*

$$\begin{aligned} \tfrac{1}{t}\left(y' - \tfrac{1}{t}y\right) &= \tfrac{1}{t}y' - \tfrac{1}{t^2}y \\ &= \tfrac{d}{dt}\left(\tfrac{1}{t}y\right) \end{aligned} \tag{2.3.32}$$

So this μ *is an integrating factor. Using it, we can rewrite 2.3.28 as*

$$\frac{d}{dt}\left(\frac{1}{t}y\right) = \frac{1}{t}te^{-t} = e^{-t}. \tag{2.3.33}$$

So

$$\frac{1}{t}y = \int e^{-t}dt = -e^{-t} + C \tag{2.3.34}$$

Multiplying by t gives

$$y(t) = -te^{-t} + Ct \tag{2.3.35}$$

A quick look at the solution $y(t) = -te^{-t} + Ct$ reveals some interesting behavior. No matter what choice is made for the constant C, we always find that $y(0) = 0$. Moreover, attempting to solve for an initial condition $y(0) = y_0$ leads to

$$y_0 = y(0) = -(0)e^0 + C(0) = 0 \tag{2.3.36}$$

The only initial condition we can ever satisfy when $t_0 = 0$ is $y_0 = 0$. Looking back at the equation 2.3.28, this should not be surprising. (Why not?)

We'll have more to say about this differential equation and it's solutions a few lectures hence.

2.3.3 More Mixing Problems

2.4 Lecture 5. Autonomous Differential Equations and Population Modeling

Differential equations describe the behavior of quantities that are changing. In order to model a particular quantity with a differential equations, one must do several things. First, the various effects that cause the quantity must be identified. Then the particular way in which they affect the quantity must be determined. Finally, these ideas must be described mathematically.

The construction of a mathematical model can be divided into three main steps:

1. **Science.** Determine the assumptions on which the model will be based. Describe the relationships among the various quantities involved.

2. **Notation.** Determine the important quantities from the assumptions in Step 1. Assign variable names, choose units and determine which variables will be independent, and which will be dependent.

3. **Mathematics.** Translate the assumptions of Step 1 into a mathematical expression involving the variables chosen in Step 2.

For us, the primary interest will be in Steps 2 and 3. The assumptions from Step 1 will usually be given in the statement of a problem, though we will at times be interested in modifying the assumptions to reflect some change to the situation being modeled.

Of course, since this is a course in differential equations, we will be primarily interested in situations where the mathematical statement from Step 3 is a differential equation.

Example 12 (Rabbit Island). *A population of rabbits lives on an island with. The island provides nutritious food for the rabbits. The rabbits eat, and do whatever else rabbits do. The population tends to increase. The rate of increase is proportional to the number of female rabbits which is in turn proportional to the size of the population.*

1. ***Science.***

    - *The rate of growth is proportional to the size of the population*

2. ***Notation.*** *Let t represent the amount of time that has passed since the discovery of Rabbit Island, measured in years. Let $R(t)$ denote the size of the rabbit population at time t. Let k represent the constant of proportionality between the size of the population and the rate of growth.*

3. ***Mathematics.***

$$\frac{dR}{dt} = kR \tag{2.4.1}$$

We've seen this differential equation before, and we know the solutions are of the form

$$R(t) = Ce^{kt}. \tag{2.4.2}$$

(The equation is both linear and separable, so either of the two analytic methods we've studied can be applied.)

If the initial population consists of 50 rabbits, then

$$50 = R(0) = Ce^0 = C \tag{2.4.3}$$

So the appropriate solution is $R(t) = 50e^{kt}$.

In order to determine the constant k, we can either make a detailed investigation of the reproductive biology of rabbits, or we can wait and gather empirical evidence. If, after one year, the population has grown to 120 rabbits, then

$$120 = R(1) = 50e^k \tag{2.4.4}$$

and solving for k produces $k = \ln\left(\frac{12}{5}\right)$, and so

$$R(t) = 50e^{\ln\left(\frac{12}{5}\right)t}. \tag{2.4.5}$$

If another rabbit island is discovered, with similar climate, etc., we would expect the growth rate k to be the same. If the initial population of the hypothetical other rabbit island were 35 rabbits, we would expect the size of the population to be

$$R(t) = 35e^{\ln\left(\frac{12}{5}\right)t}. \tag{2.4.6}$$

After one year, there would be $R(1) = 35e^{\ln\left(\frac{12}{5}\right)} = 35\frac{12}{5} = 84$ rabbits.

We can also study this differential equation using qualitative methods. One interesting thing about this differential equation, is that the RHS does not depend on the independent variable. Consequently, if we draw the direction field, the direction field elements will have constant slope along horizontal lines. This can be seen in Figure 2.4.

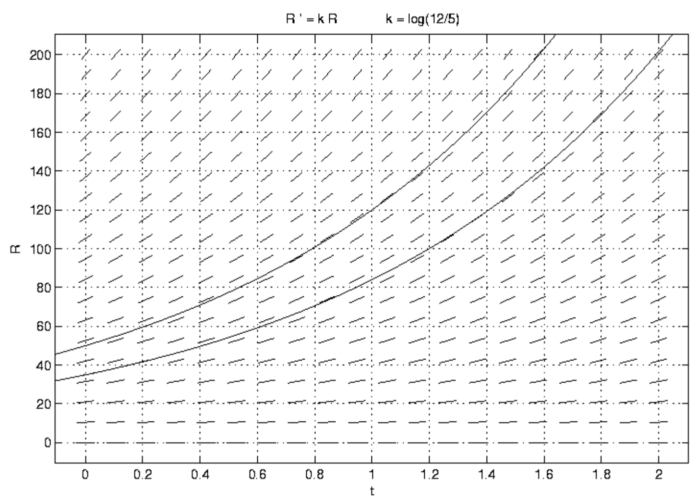

Figure 2.5: Direction field for $R' = \ln\left(\frac{12}{5}\right) R$.

Because of this property, any vertical slice of the tR-plane contains all the information of the whole plane. We can imagine squishing the direction field in Figure 2.4 onto the R-axis. Doing so produces what is called the phase line.

[PICTURE OF PHASE LINE HERE]

Any differential equation of the form

$$\frac{dy}{dx} = f(y) \tag{2.4.7}$$

has the property that direction field elements will have constant slope along horizontal lines. Consequently, a phase line can be drawn for any such equation. Such differential equations are *autonomous.*

The model developed for our study of Rabbit Island predicts that, given enough time, the population will grow to any given size, no matter how large. This seems unreasonable. We may want to modify our assumptions.

Example 13 (Rabbit Island II)**.** *1.* ***Science.***

- *The rate of growth is proportional to the size of the population.*
- *Scarcity of resources tends to limit this growth as the size of the population increases.*

2. ***Notation.*** *Let* t*,* $R(t)$ *and* k *be as they were before. Let* h *be a constant that represents the strength of the environmental effects limiting the growth.*

3. ***Mathematics.*** *We wish, as before, to get a differential equation whose LHS is* $\frac{dR}{dt}$*. On the RHS, we may expect to get a term for each assumption.*

The first assumption, as before, leads to kR*. For the second assumption, we would like a term that is negligible when the population is small, but becomes a large (negative) number when the population is bigger. There are many possible choices, so lets try to choose a simple one:* hR^2*. This has the advantage of being a polynomial, just like the first term. Thus we get*

$$\frac{dR}{dt} = kR - hR^2. \tag{2.4.8}$$

This equation can be rewritten in the form

$$\frac{dR}{dt} = kR(1 - \frac{hR}{k}) = kR(1 - \frac{R}{M}), \tag{2.4.9}$$

where $M = k/r$*. This model is commonly referred to as the* logistic model*.* M *represents the largest sustainable population, or* carrying capacity*, for the system. This can be seen by constructing the phase line.*

[PHASE LINE PICTURES HERE]

Compare these to the [DIRECTION FIELD] and the [SOLUTION CURVES].

2.4.1 Summary of Autonomous Equations and the Phase Line

For a differential equation

$$\frac{dy}{dt} = f(y) \tag{2.4.10}$$

1. Sketch the graph of $f(y)$ vs. y. Note where the graph is positive, negative and equal to 0.

2. Draw the phase line: Equilibrium points occur wherever $f(y) = 0$, solutions are increasing wherever $f(y) > 0$, and decreasing wherever $f(y) < 0$.

3. If desired, the phase line may now be used to sketch soltuions.

2.5 Population Models

We have looked at two different population models, the exponential model

$$\frac{dP}{dt} = kP, \tag{2.5.1}$$

and the logistic model

$$\frac{dP}{dt} = kP - hP^2 = kP(1 - \frac{P}{M}). \tag{2.5.2}$$

These models are both autonomous equations. Not all populations must be so, however. The differential equation

$$\frac{dy}{dt} = r(t)y - k \tag{2.5.3}$$

might also be used to model the growth of a population.

When $k = 0$, 2.5.3 looks similar to the exponential model, with the growth constant k replaced by the function $r(t)$. This could be interpreted as a growth constant the varies with time. Perhaps $r(t)$ is larger in the summer and smaller in the winter, for instance.

Positive values of k could be interpreted as hunting or harvesting of resources at a constant rate.

Let's take $r(t) = \frac{1}{5}(1+sin(t))$ and $k = \frac{1}{5}$. Some solution curves are shown in figure 2.5. Note that while the size of the population fluctuates, initial populations that are sufficiently large (larger than 0.83 or so) show a net increase in population. Smaller populations show a net decrease, eventually falling to 0, indicating that the population has been wiped out.

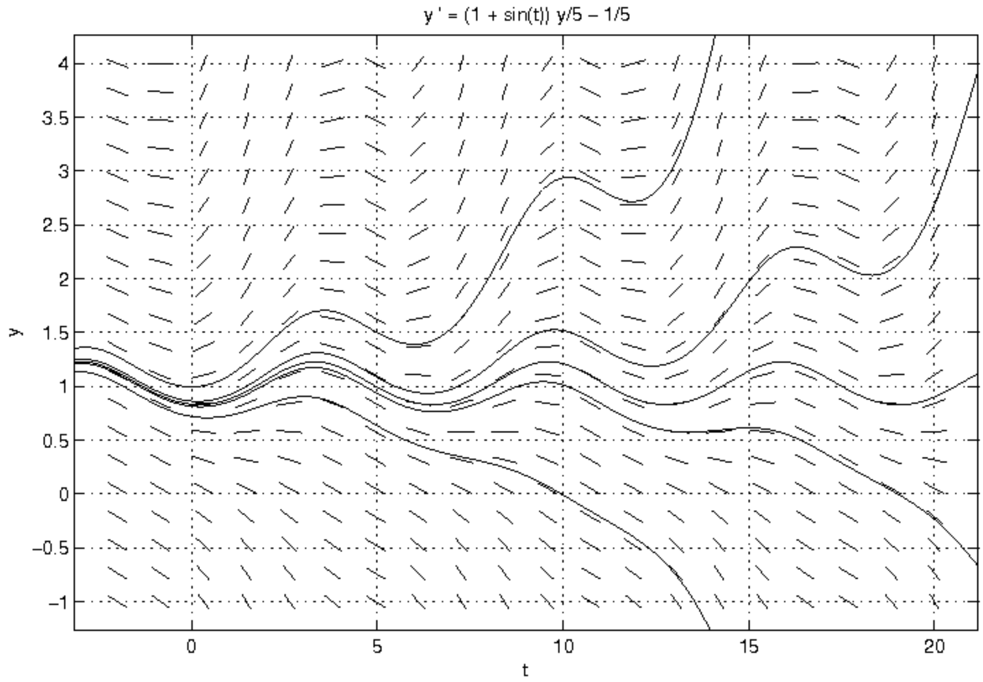

Figure 2.6: Direction field for $y' = \frac{1}{5}(1 + sin(t))y - \frac{1}{5}$.

We could modify this "seasonal model" to create a "seasonal logistic model", by replacing the harvesting term with a term that represents environmental constraints on the population. For example

$$\frac{dy}{dt} = \frac{(1 + sin(t))}{5}y - hy^2 \tag{2.5.4}$$

Figure 2.5 shows the result when $h = 0.1$. Notice that as with the standard logistic model, solutions with small initial conditions tend to increase toward some stable solution, while solutions with a large initial condition decrease toward that stable solution.

2.6 Lecture 6. Existence and Uniqueness

[Examples]

It is important to know, when using qualitative or numerical techniques, whether the solutions you are attempting to study actually *exist*. It is also important, when using any technique, to know if initial value problems have *unique* solutions.

The theory of existence and uniqueness is a bit simpler in the case of Linear equations, so that is where we will begin.

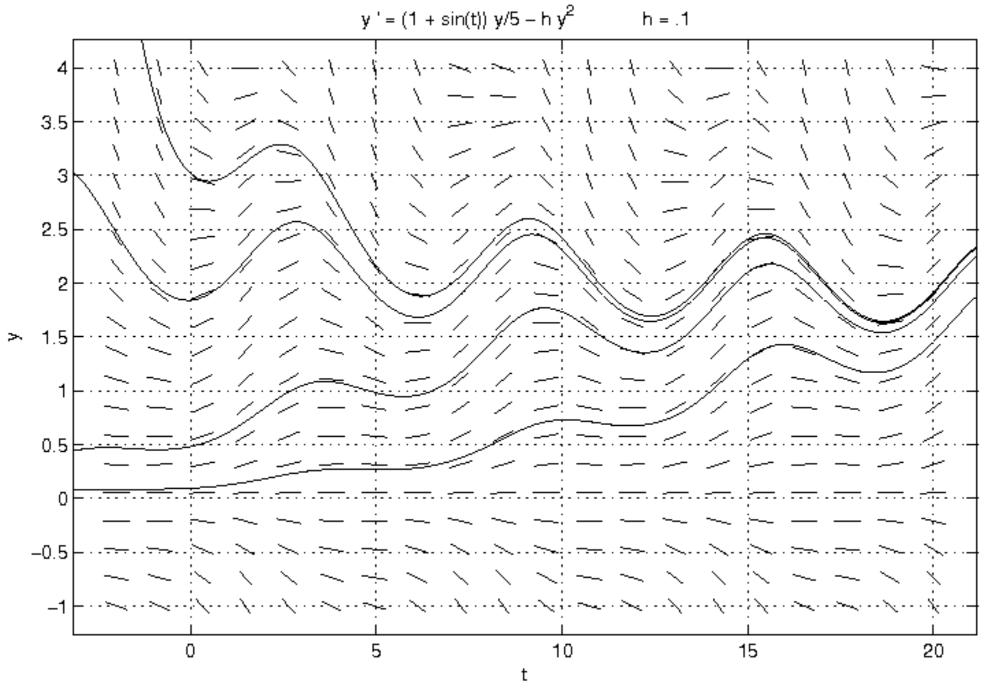

Figure 2.7: Direction field for $y' = \frac{1}{5}(1 + sin(t))y - .1y^2$.

2.6.1 Existence and Uniqueness for Linear Differential Equations

Theorem 1 (Existence and Uniqueness for Linear Differential Equations)**.** *If the functions $p(t)$ and $q(t)$ are continuous on an open interval $I = (\alpha, \beta)$ containing the point $t = t_0$ and t_0 is any real number, then there exists a unique solution $y = \phi(t)$ to the initial value problem*

$$y' + p(t)y = g(t), \quad y(t_0) = y_0 \tag{2.6.1}$$

The The solution may be defined on all of the interval $I = (\alpha, \beta)$.

Example 14. *Consider*

$$y' - \frac{1}{t}y = te^{-t} \tag{2.6.2}$$

In this case $p(t) = -\frac{1}{t}$ and $g(t) = te^{-t}$. The function g is continuous for all values of t, but p is discontinuous at $t = 0$. Taking the interval $I = (\alpha, \beta) = (0, \infty)$, we see that both p and g are continuous on all of I. The theorem tells us that for any $t_0 \in (0, \infty)$ and any $y_0 \in \mathbb{R}$, there is a solution to the differential equation satisfying

$$y(t_o) = y_0 \tag{2.6.3}$$

and that this solution is defined for all values $0 < t < \infty$. *Moreover, on this interval, there is* no other *solution satisfying that initial condition.*

We found the general solution in a previous lecture:

$$y(t) = -te^{-t} + ct. \tag{2.6.4}$$

So if

$$y_0 = y(t_0) = -t)0e^{-t_0} + ct_0, \tag{2.6.5}$$

then

$$c = \frac{y_0 + t_0 e^{-t_0}}{t_0}, \tag{2.6.6}$$

and

$$y(t) = -te^{-t} + \left(\frac{y_0 + t_0 e^{-t_0}}{t_0}\right) t \tag{2.6.7}$$

is the only *solution to equation 2.6.2.*

Example 15. *In example 14, the same reasoning holds if* $t_0 \in (-\infty, 0)$. *But what if* $t_0 = 0$*?*

Since $p(t) = -\frac{1}{t}$ *is not continuous at* $t = 0$, *we cannot apply Theorem 1 to conclude that unique solutions exist. Can we conclude that solutions do not exist? Or that solutions exist, but are not unique? In general, no. We can't reach either of these conclusions without investigating the situation more deeply.*

in the case of equation 2.6.2, the initial value problem

$$y' - \frac{1}{t}y = te^{-t}, \quad y(0) = y_0 \tag{2.6.8}$$

has no solution if $y_0 \neq 0$. *This can be see by substituting in to the differential equation (we can multiply by* t *first to get* $ty' - y = t^2 e^{-t}$*) to get*

$$0y' - y_0 = 0^2 e^0. \tag{2.6.9}$$

The only way this can be true is if $y_0 = 0$. *On the other hand, when* $y_0 = 0$ *the equation has many solutions. This can be seen by simply looking at the solutions we've already found:*

$$y(t) = -te^{-t} + ct = t(c - e^{-t}). \tag{2.6.10}$$

This satisfies $y(0) = 0$ *for any choice of* c.

2.6.2 Existence and Uniqueness for Nonlinear Differential Equations

In order to discuss existence and uniqueness in the case of non-linear differential equations, is is necessary to introduce the idea of *partial derivatives.* While there is a great deal that can be said about partial derivatives, this is all we need to know: it is possible to differentiate a function f of several variables $x_1, \ldots, x_n$ with respect to one of the variables x_i by treating all the other variables as constants. The result is the partial derivative of f with respect to x_i, written

$$\frac{\partial f}{\partial x_i} \tag{2.6.11}$$

Example 16. *If* $f(x, y) = 2x + 3xy - 4y^2 + 5e^{x^2 y}$, *then*

$$\frac{\partial f}{\partial x} = 2 + 3y - 0 + 10xye^{x^2 y} \tag{2.6.12}$$

and

$$\frac{\partial f}{\partial y} = 0 + 3x - 8y + 5x^2 e^{x^2 y}. \tag{2.6.13}$$

Now we can state

Theorem 2 (Existence and Uniqueness for Nonlinear Differential Equations)**.** *Suppose that* $f(t, y)$ *and* $\frac{\partial f}{\partial t}$ *are continuous in a rectangle* R *is the* ty*-plane defined by* $\alpha < t < \beta$ *and* $\gamma < y < \delta$, *and that the point* (t_0, y_0) *is in* R. *Then there is a unique solution* $y = \phi(t)$ *to the initial value problem*

$$y' = f(t, y), \quad y(t_0) = y_0 \tag{2.6.14}$$

defined on some interval $(t_0 - h, t_0 + h)$, *contained in* (α, β).

Note that the solution may only be defined on a very small interval. This is a notable change from the case of linear differential equations. (In fact the solution will be defined for as long as it remains in the rectangle R. The problem is that it may leave through the top or bottom very quickly.)

Example 17. *Consider the equation*

$$\frac{dy}{dt} = \frac{1}{ty}. \tag{2.6.15}$$

In this case $f(t, y) = \frac{1}{ty}$ *is discontinuous only along the lines* $t = 0$ *and* $y = 0$. *Similarly,* $\frac{\partial f}{\partial y} = -\frac{1}{ty^2}$ *fails to be continuous only along the lines* $t = 0$ *and* $y = 0$.

In this case, if $(t_0, y_0) = (3, 3)$ *for instance, we could choose the rectangle* R *to consist of all* (t, y) *such that* $1 < t < 4$ *and* $2 < y < 5$. *(There are,* *of course, many other possible choices.) Equation 2.6.15 will have a unique solution satisfying* $y(3) = 3$*, and that solution will be defined for at least as long as it remains in the rectangle* R*.*

Since equation 2.6.15 is separable, we can find solutions by integrating

$$\int y dy = \int \frac{dt}{t} \tag{2.6.16}$$

to get

$$y^2 = ln|t| + c, \tag{2.6.17}$$

or

$$y(t) = \pm\sqrt{ln|t| + c} \tag{2.6.18}$$

Notice that the solution is not defined at any point along the line $t = 0$*, and behave strangely (has a vertical asymptote, and two possible solutions (*$\pm$*)) along the line* $y = 0$*.*

2.6.3 Implications for Graphs of Solutions

2.7 Lecture 7. Numerical Technique: Euler's Method