21-124 MODELING WITH DIFFERENTIAL EQUATIONS

LECTURE 7: AUTONOMOUS SYSTEMS AND THE PREDATOR-PREY EQUATIONS

1. Autonomous Systems of Differential Equations

1.1. **General Form.** The general form of a (two-dimensional) autonomous system is

$$\left.\begin{aligned} \tfrac{dx}{dt} &= f(x,y) \\ \tfrac{dy}{dt} &= g(x,y) \end{aligned}\right\} \tag{1.1}$$

It is *autonomous* because the functions f and g do not depend on the independent variable. It is *two-dimensional* since the system has two dependent variables.

We can write this pair of equations as a single vector equation:

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

If we introduce the vector $Y = \begin{bmatrix} x \\ y \end{bmatrix}$, then we can write this as

$$\frac{dY}{dt} = F(Y). \tag{1.2}$$

1.2. **Solutions.** A solution to the system 1.1 is a pair of functions, $x(t)$ and $y(t)$, such that

$$\begin{aligned} x'(t) &= f(x(t), y(t)) \\ y'(t) &= g(x(t), y(t)). \end{aligned}$$

For example, it is a straightforward matter to check that the system

$$\left.\begin{aligned} \tfrac{dx}{dt} &= y \\ \tfrac{dy}{dt} &= -x \end{aligned}\right\}$$

has as a solution the pair $x(t) = \sin(t)$, $y(t) = \cos(t)$. To verify this, one need only compute $x'(t) = \cos(t) = y(t)$ and $y'(t) = -\sin(t) = -x(t)$.

A solution to the equation 1.2 is a *vector valued function*

$$Y(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

Such that $\frac{dY}{dt}(t) = F(Y(t))$. The functions $x(t)$ and $y(t)$ are the same as in the solution to the system 1.1. The difference is merely one of notation and point of view. In the first case, we view the functions as two separate objects ($x(t)$ and $y(t)$) that work together, while in the second case, we view the solution as a single object ($Y(t)$) composed of two pieces ($x(t)$ and $y(t)$).

1.3. **Existence and Uniqueness.** The statement about existence and uniqueness parallels the general existence and uniqueness we discussed in 21-260 (Edwards and Penney, p. 23). The hypotheses in both cases relate to the functions on the right hand side of the equations, and the derivatives of those functions with respect to the dependent variables.

Theorem 1 (Existence and Uniqueness). *If f, g, $\frac{df}{dx}$, $\frac{df}{dy}$, $\frac{dg}{dx}$ and $\frac{dg}{dy}$ are continuous in a rectangle R in the xy-plane containing the point (a,b), there there is a solution satisfying the initial conditions $x(0)=a$, $y(0)=b$.*

Moreover this solution is unique for as long as it remains in the rectangle R.

The main reason we are interested in this theorem is that, provided the hypotheses are satisfied, a solution curve represents a barrier in the phase plane (xy-plane), in much the same way that an equilibrium point of an ordinary first order differential equation represents a barrier in the phase line for that equation. If Uniqueness holds, then solution curves can not cross.

For example, if a solution curve of a closed system is periodic (forms a closed orbit), then solutions with initial conditions inside the loop must remain forever inside the loop.

2. The Predator-Prey Equations

The predator-prey model is based on the following assumptions:

- In the absence of predators, the prey population increases at a rate proportional to its size.
- The size of the prey population is reduced at a rate proportional to the number of interactions between the predators and the prey. (We take this rate to be the size of the predator population times the size of the prey population.)
- The size of the predator population increases at a rate proportional to the number of interactions between the predators and the prey.
- In the absence of prey, the predator population decreases at a rate proportional to its size.

If R is the size of the prey population (rabbits) and F is the size of the predator population (foxes), then the mathematical model is:

$$\frac{dR}{dt} = \alpha R - \beta RF$$
$$\frac{dF}{dt} = -\gamma F + \delta RF$$

Just as we can use dfield to study a single differential equation, the MATLAB routine `pplane` can be used to study a system of two differential equations. You will have to download `pplane6.m` and `ppn6out.m` from the same site you got `dfield5.m`. (You can use the version for MATLAB 6.0 if that is what you are using. These notes will assume you are using version 5.)

Once you have downloaded the files, type `pplane5` at a MATLAB prompt. If you have used dfield in the past, then `pplane` will seem familiar. The interface is quite similar.

After you enter the equations for x' and y' and click "proceed", `pplane` will open a new window showing a coordinate grid with axes labeled x and y. This is the "phase plane" for the system. On this plane, the vector field for the equations you entered is shown. This vector field shows the direction that solution curves will travel. You should read pp. 375 to 377 in Edwards and Penney to learn more about this vector field and the phase plane.

Using `pplane`, we can investigate the Predator-Prey system

$$\begin{aligned} \frac{dR}{dt} &= .2R - .03RF \\ \frac{dF}{dt} &= -.3F + .01RF \end{aligned}$$

We find that the solution curves form (apparently) closed loops. We can follow the solution around one of the loops, interpreting how the populations are affecting each other along the way. Let's begin at the point where the rabbit population reaches it's maximum.

Here the fox population is moderate in size. This is a good situation for the foxes. There are plenty of rabbits, so food is easy to find. At the same time there are not too many foxes, so there is not much competition for the rabbits. Thus the fox population is able to increase it's size. At the same time, the foxes are depleting the rabbit population, which decreases in size.

By the time the fox population has reached it's maximum size, the rabbit population has been severely reduced. The large number of foxes can no longer find enough game to sustain itself, and the fox population begins to decrease. There are still a large number of foxes, though, and they continue to eat the rabbits at a rate that exceeds the rabbit's birth rate. For the next portion of the cycle, both populations decrease.

Eventually, the fox population decreases to a point where there are no longer enough of them to keep the rabbit population in check. The rabbit population begins to increase. As it does so, the fox population continues to decrease.

Once the rabbit population recovers sufficiently, the small fox population can now find the food that it needs to survive and grow. For the next portion of the cycle, both populations increase, eventually returning us to our starting point.

Since the solutions form closed loops, they divide the plane into two regions. If you click an initial condition inside a solution curve, MATLAB computes a new solution, which is a smaller cycle contained inside the first. Repeating this produces a series of "concentric" loops, which seem to focus onto a single point. In fact this is an *equilibrium point* for the system.

Recall that for the first order autonomous equation,

$$\frac{dy}{dt} = h(y)$$

the equilibrium points were computed by solving $\frac{dy}{dt} = 0$ or $h(y) = 0$. In the case of a system, in vector notation, we must have $\frac{dY}{dt} = 0$ or

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) = 0 \\ \frac{dy}{dt} &= g(x,y) = 0 \end{aligned}$$

In the case of our Predator-Prey Equations, this becomes

$$\begin{aligned} \frac{dR}{dt} &= R(\alpha - \beta F) = 0 \\ \frac{dF}{dt} &= F(-\gamma + \delta R) = 0. \end{aligned}$$

There are two possibilities:

$$\left\{ \begin{array}{c} R = 0 \\ F = 0 \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{c} F = 6.67 \\ R = 30 \end{array} \right\}$$

The first possibility agrees with our intuition. If we begin with no rabbits and no foxes, we never expect to have any in the future. The second case, on the other hand, represents the situation where the two populations are in exact balance.