21-124 MODELING WITH DIFFERENTIAL EQUATIONS

LECTURE 13: NULLCLINES
EXPERIMENTS AND DISCUSSION

In the twelfth lecture, we introduced the concept of *nullclines*, and showed how they could be used to predict the behavior of a system. Today, we will apply these concepts to some examples.

1. Consider the system of differential equations

$$\begin{aligned} \tfrac{dx}{dt} &= x - y - x^3 \\ \tfrac{dy}{dt} &= -x. \end{aligned}$$

   (a) Find the nullclines for the system, and sketch their graphs in the phase plane. Identify any equilibrium points.
   (b) Determine the direction solutions must travel as they cross the nullclines, and in each of the regions determined by the nullclines.
   (c) Sketch a representative sample of solution curves.
   (d) What, if anything, have you determined about the equilibrium point(s) you found?
   (e) check your results using `pplane6`.

2. Repeat the steps above for the system

$$\begin{aligned} \frac{dx}{dt} &= x - y - x^3 \\ \frac{dy}{dt} &= x. \end{aligned}$$

3. Repeat the steps above for the system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -\sin(x) - y. \end{aligned}$$

4. Consider the predator-prey system

$$\begin{aligned} \frac{dR}{dt} &= 8R - 2RF \\ \frac{dF}{dt} &= -6F + 3RF. \end{aligned}$$

(a) Find the nullclines for this system and sketch them in the phase plane. Find any equilibrium points. Determine the directions that solutions must travel and sketch some solution curves.
(b) Modify the system to allow for harvesting of both the predators and prey at a rate proportional to the size of the populations (with the same constant of proportionality for both predators and prey).
(c) Find the nullclines for the modified system and sketch them in the phase plane. Find any equilibrium points. Determine the directions that solutions must travel and sketch some solution curves.
(d) Which population has more to fear from this harvesting? Why?

5. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= y - x(x^2 + y^2 - 1) \\ \frac{dy}{dt} &= -x - y(x^2 + y^2 - 1). \end{aligned}$$

(a) Show that the origin is an equilibrium point.
(b) Find the linearization for the system at the origin.
(c) Classify the equilibrium point for the linearized system. What, if anything does this tell you about the equilibrium point at the origin of the non-linear system?
(d) Show that if $x(t)^2 + x(t)^2 < 1$, then $\frac{d}{dt}[x(t)^2 + x(t)^2] \geq 0$. What does this tell you about the equilibrium point at the origin?