21-124 MODELING WITH DIFFERENTIAL EQUATIONS

LECTURE 10: A MASS–SPRING SYSTEM IN TWO DIMENSIONS

In Lecture 10 we will build a mathematical model for a mass on a spring that is free to move in a vertical plane. The spring has a spring constant of $k = .7$. One end is fixed at the origin, the other end is attached to a mass with $m = 1.2$. The equilibrium length of the spring is 1, so that if the mass is inside the unit circle, the spring pushes it away from the origin, while if it is farther away, it is pulled toward the origin.

1. The Basic Setup

The force of gravity pulls downward with a constant strength of mg, where $g = 9.8$. Consequently, the force of gravity can be represented by the vector

$$F_g = -mg \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The force due to the spring is a bit more difficult to analyze. If the equilibrium position were at the origin, then the force would be given by

$$-k \begin{bmatrix} x \\ y \end{bmatrix},$$

$-k$ times the vector pointing from the equilibrium point to the position of the mass. Since the spring is in equilibrium on the unit circle, we must subtract the unit vector pointing in the $[x, y]^T$ direction. Doing so we get

$$F_s = -k\left(\begin{bmatrix} x \\ y \end{bmatrix} - \frac{1}{\sqrt{x^2+y^2}}\begin{bmatrix} x \\ y \end{bmatrix}\right).$$

The vector in parentheses points to the position of the mass from the point on the unit circle closest to the mass. So the total force acting on the mass is

$$F = -mg \begin{bmatrix} 0 \\ 1 \end{bmatrix} - k\left(\begin{bmatrix} x \\ y \end{bmatrix} - \frac{1}{\sqrt{x^2+y^2}}\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

Now, since Newton's law tells us that Force equals Mass times Acceleration, we can write

$$F = m \begin{bmatrix} \frac{d^2x}{dt^2} \\ \frac{d^2y}{dt^2} \end{bmatrix}.$$

Equating these two expressions for the force, we get a pair of second order equations:

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\frac{k}{m}\left(1 - \frac{1}{\sqrt{x^2+y^2}}\right)x \\ \frac{d^2y}{dt^2} &= -g - \frac{k}{m}\left(1 - \frac{1}{\sqrt{x^2+y^2}}\right)y \end{aligned}$$

In order to use MATLAB's numerical solvers, we must convert this system of second order equations to a system of first order equations. We do this by making the definitions $\frac{dx}{dt} = v$ and $\frac{dy}{dt} = w$. Then

$$\begin{aligned} \frac{dv}{dt} &= -\frac{k}{m}\left(1 - \frac{1}{\sqrt{x^2+y^2}}\right)x \\ \frac{dw}{dt} &= -g - \frac{k}{m}\left(1 - \frac{1}{\sqrt{x^2+y^2}}\right)y \end{aligned}$$

The first order system we want to study is thus

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= -\frac{k}{m}\left(1 - \frac{1}{\sqrt{x^2+y^2}}\right)x \\ \frac{dy}{dt} &= w \\ \frac{dw}{dt} &= -g - \frac{k}{m}\left(1 - \frac{1}{\sqrt{x^2+y^2}}\right)y \end{aligned}$$

An M-file for this system (`twodspring.m`) is available from the 21-124 website.

We can experiment with different initial conditions for the system. The most convenient way to view and interpret the solutions is to look at the projection of solutions onto the xy-plane. Then we see the path the mass would follow in physical space. The mass seems to bob up and down, just as one might expect.

We can determine algebraically that this system has a single equilibrium point at $[0, -17.8]^T$. Notice that the motion of solutions tends to center around this point.

2. A Different Point of Attachment

What if the fixed end of the spring is attached at a point $[a, b]^T$, rather than the origin? Then rather than the vector

$$\begin{bmatrix} x \\ y \end{bmatrix},$$

which points from the origin to the position of the mass, we must consider the vector

$$\begin{bmatrix} x-a \\ y-b \end{bmatrix},$$

which points from $[a, b]^T$ to the position of the mass. Then

$$F = -mg\begin{bmatrix} 0 \\ 1 \end{bmatrix} - k\left(\begin{bmatrix} x-a \\ y-b \end{bmatrix} - \frac{1}{\sqrt{(x-a)^2+(y-b)^2}}\begin{bmatrix} x-a \\ y-b \end{bmatrix}\right)$$

The corresponding system of first order equations is then

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= -\frac{k}{m}\left(1 - \frac{1}{\sqrt{(x-a)^2+(y-b)^2}}\right)(x-a) \\ \frac{dy}{dt} &= w \\ \frac{dw}{dt} &= -g - \frac{k}{m}\left(1 - \frac{1}{\sqrt{(x-a)^2+(y-b)^2}}\right)(y-b). \end{aligned}$$

This change hardly seems worth making, since it simply translates the same behavior we had before to a different part of the plane. What we gain, however, is the possibility that a and b may be functions of t. That is instead of *fixing* one end of the spring, we can move it around the plane.

3. Moving the Spring

We will now consider the case where the spring moves in a small circle around the origin. Specifically, $a(t) = .1\cos(\omega t)$ and $b(t) = .1\sin(\omega t)$. The M-file **movingspring.m**, available from the 21-124 website, can be used to model this situation. It uses a global variable, **omega**, which allows us to easily study different forcing frequencies.

Using $\omega = 1$ or $\omega = .1$ and allowing the mass to start from rest at $x = 0$ and $y = -17.8$ we see that the resulting oscillations are relatively small (with amplitude on the order of .5). An interesting frequency to try is $\omega_0 = \sqrt{\frac{k}{m}} = .7638$, the natural frequency for a one dimensional system with the same mass and spring constant. When we do so, we find that the oscillations are much larger.

Next the next lecture we will investigate in more detail the response of this system at varying forcing frequencies.