21-124 MODELING WITH DIFFERENTIAL EQUATIONS

LECTURE 1: NAVIGATING IN UNIX AND BUILDING MATHEMATICAL MODELS

In the first lecture, we began with a brief look at navigating within the UNIX operating system. We then discussed the process of building a mathematical model and looked at several examples of mathematical models.

1. Navigating in UNIX

CMU only supports MATLAB on its UNIX and LINUX systems, so I will assume you are using a system of that type (there is essentially no difference for our purposes). I will also assume you have logged onto your account.

To begin, you need to have an Xterm window open. If you have one, great. If not, you will have to open one. The procedure for doing so varys a bit depending on how your account is set up, but it goes something like this: move the cursor to an empty portion of the desktop (i.e. where there are no windows), then try clicking with the mouse buttons. One of them, probably the right one will give you a list of applications, including "xterm". Use the mouse to select that option.

Now, in order to use the other programs we need, you simply have to type their name on the command line of your xterm window, and hit `[enter]`. For instance, entering `netscape` will bring up a Netscape window which you can use to browse the web. (The homepage for this course is http://www.math.cmu.edu/~21124/.)

Notice that while Netscape is running, you can't enter any more commands in the xterm window. You can solve this problem by entering `netscape &`, instead of simply `netscape`. The `&` makes Netscape run in the background (from xterm's point of view – you can still use it). If you forget the `&`, place the cursor in the xterm window and type `[ctrl]-z`, and then enter `bg` on the command line.

Another application we will be using is emacs. Again, you can open an emacs window by entering `emacs` or `emacs &` on the command line. Emacs is a text editor. It is similar to a word processor but lacks any formating or desktop publishing features. Emacs is pretty easy to navigate using the pull–down menus. If you would like more detail about using emacs, you can follow the link on the course website to "A beginners guide to using emacs".

The application we will use most often is MATLAB. This can be accessed by entering `matlab` on your xterm command line. Since MATLAB runs within the xterm window, you should not run it in the background (i.e. do not use an `&`).

We will talk a great deal about using MATLAB in the weeks ahead.

2. The Modeling Process

The study of differential equations is, in some sense, about predicting the future. If I have a contraption made of springs and levers and I kick it just so, I want to know what it is going to do.

Differential equations are not just used in the physical sciences. They are used in ecology and biology to study the growth of populations and ecosystems. They are also used in economics to model the behavior of businesses and investments.

Derivatives describe rates of change, and so anywhere something is changing in a quantifiable way, a differential equation (or collection of differential equations) can be used to study those changes. Constructing an appropriate mathematical model is a three stage process:

1. **Science:** Determine the assumptions on which the model will be based. Describe the relationships among the quantities involved.
2. **Notation:** Determine the important quantities in the assumptions from Step 1. Assign variable names to these quantities and determine units for the variables.
3. **Mathematics:** Translate the assumptions from Step 1 into equations involving the variables in Step 2.

Since we are interested in equations involving derivatives, we should be on the lookout for some key words: rate of change, velocity, acceleration, increasing, decreasing, growth, decay, and any other words that describe *changing* quantities.

3. Examples: Population Growth

Let's look at an example of how a mathematical model is constructed. We will try to model the growth of a population of dandelions in a field.

Suppose it takes a dandelion a month to grow to maturity and produce seeds. (I just made that up, but it seems about right.) If each dandelion produces about the same number of seeds, and the same percentage of seeds from each dandelion survives and grows to maturity, then the number of dandelions one month from now will be proportional to the number of dandelions we have today. The growth of the population is proportional to the current size of the population.

3.1. **The Exponential Growth Model. Step 1:** • The rate of growth of the population is proportional to the size of the population.

Now, what are the important quantities? Well, the size of the population, for one. Also time. We will also need a constant of proportionality to represent the relationship between the size of the population and the rate of growth.

Step 2: • Let P represent the size of the population. If our field is big enough, but not too big, it makes sense for P to measure the number of dandelions. This does introduce some problems. The solutions to a differential equation will be continuous functions, but the number of dandelions should always be an integer. One solution is to suppose that P approximates the number of dandelions. Another solution might be to let P denote the *mass* of dandelions (in grams or kilograms).

• Let t denote the time that has passed. It seems appropriate to measure time in months.

• Let k denote the constant of proportionality between the rate of growth of the population and the size of the population.

Now, if $P(t)$ is the size of the population at time t, the *rate of growth* of the population is the derivative $P'(t) = \frac{dP}{dt}$. So...

Step 3: • The mathematical model is

$$\frac{dP}{dt} = kP.$$

This is a separable equation, and so we can find the general solution without too much difficulty:

$$P_g(t) = Ce^{kt}.$$

Because the solutions of the differential equation are exponential functions, this is called the *Exponential Growth Model.* There are still some things we don't know, though. There are two undetermined constants. Determining them is our next order of business.

Suppose at the start of our experiment, we look out and there are 50 dandelions in our field. Well, then we must have $P(0) = 50$. But

$$\begin{aligned} 50 &= P(0) \\ &= Ce^{k(0)} \\ &= C(1), \end{aligned}$$

and so we must have $C = 50$, and $P(t) = 50e^{kt}$.

Now suppose that a month later, we see that there are 120 dandelions in the field. Well, in that case, $P(1) = 120$, so

$$\begin{aligned} 120 &= P(1) \\ &= 50e^{k(1)} \\ &= 50e^{k}. \end{aligned}$$

It follows that $e^k = \frac{120}{50}$ and hence $k = \ln\left(\frac{120}{50}\right) = \ln\left(\frac{12}{5}\right)$. So

$$P(t) = 50e^{\ln\left(\frac{12}{5}\right)t}.$$

The two observation of $P(0)$ and $P(1)$ allowed us to determine the two constants C and k. This model indicates that the growth of the dandelion population is unlimited. There are some problems with this conclusion. For instance, if N is the number of particles in the universe (about 10^{80}) the model predicts that at some time in the future, there will be more than N dandelions in the field.

Nonetheless, this model does work well for small populations. Thus we are inclined to *modify* the model by adding an additional assumption.

3.2. **The Logistic Growth Model. Step 1:** • As before, reproductive forces cause the population to grow at a rate proportional to the size of the population.

• A scarcity of resources (minerals, water, etc.) reduces this growth rate. This effect is small when the population is small, but grows quickly as the population increases.

Step 2: • We still require the variables P, t and k, as before.

• Now we also require a constant to indicate the relative effect of scarcity on the population. We will use h.

Step 3: • When a rate of change is affected by multiple sources, it is often possible to get a net effect by simply adding the individual effects. So, our equation for

$\frac{dP}{dt}$ will have one term, kP, derived from the first assumption, and a second term derived from the second assumption.

The second term will be of the form $-hf(P)$, where $f(P)$ is a small when P is small, but increases quickly as P increases. The simplest choice that comes to mind is P^2. I is usually best to begin with the easiest possibility, and try more complicated options only if the simple ones do not work well.

With the choice we have made, our model becomes

$$\frac{dP}{dt} = kP - hP^2.$$

We can re-write this in the form

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right),$$

where $M = \frac{k}{h}$ represents the largest sustainable population. This model is known as the *Logistic Growth Model.*

4. Examples: Toricelli's Law

Recall that Toricelli's Law describes the rate at which water will flow from a tank with a hole at the bottom. This velocity is $v = \sqrt{2gy}$, where g is the gravitational acceleration, and y is the depth of the water.

We see then that if V is the volume of water in the tank, then

$$\frac{dV}{dt} = -av = -a\sqrt{2g}\sqrt{y},$$

where a represents the area of the hole. By the chain rule

$$\frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt}.$$

Combining these, we can get a differential equation for the depth y:

$$\frac{dy}{dt} = -a\sqrt{2g}\frac{\sqrt{y}}{\left(\frac{dV}{dy}\right)}$$

Now, $\frac{dV}{dy}$ depends on the shape of the tank. If $A(y)$ is the surface area of the water at depth y, then we can integrate to calculate the volume when the depth is y:

$$V(y) = \int_0^y A(z)dz.$$

Applying the Fundamental Theorem of Calculus, we see that the derivative $\frac{dV}{dy}$ is simply $A(y)$. Thus we have

$$\frac{dy}{dt} = -a\sqrt{2g}\frac{\sqrt{y}}{A(y)}.$$

For example, a conical (upside down) tank can be obtained by revolving the line $y = x$ about the y-axis. Assume that the hole at the bottom (vertex) is small enough to have a negligable effect on the depth of the tank. In this case, the surface area $A(y)$ is that of a circle with radius $x = y$. so...

$$\begin{aligned}
\frac{dy}{dt} &= -a\sqrt{2g}\frac{\sqrt{y}}{A(y)} \\
&= -a\sqrt{2g}\frac{\sqrt{y}}{\pi y^2} \\
&= -\frac{a\sqrt{2g}}{\pi}\frac{\sqrt{y}}{y^2} \\
&= -\frac{a\sqrt{2g}}{\pi}y^{-\frac{3}{2}}.
\end{aligned}$$

This differential equation is separable, and easily solved by separating the variables and integrating.