21-124 MODELING WITH DIFFERENTIAL EQUATIONS

PROJECT 1

Project 1 requires you to use techniques you have learned this semester to analyze and draw conclusions about several different modeling scenarios. You should complete all three problems. You may only discuss your progress with Hassaan or myself, and all questions regarding Project 1 should be addressed to Hassaan or myself. In particular, you should not discuss any aspect of the problems with your classmates.

Project 1 is due at in at 5:00 on Tuesday, April 23. Any time before then, you can show me a rough draft of your report, which I will review and discuss with you as promptly as possible. This is not required, but I strongly recommend it.

For each problem, there are several questions you are asked to answer. If you see another way to approach the problem that is more interesting to you, you may, in consultation with me, alter the questions somewhat. If you do that, you should definitely let me review a rough draft of your report.

1. In this problem we will derive and apply the square law of military strategy. (You need not use a computer in completing this problem, but you may wish to use it to view some solutions and get an idea what is going on) We assume that two opposing forces of x_0 and y_0 forces face each other in an open field. In this setting, the effectiveness of each force is directly proportional to it's size. Thus, as the conflict unfolds, the rate of change of one army is directly proportional to the size of it's enemy. The system differential equations that describe this scenario is

$$\begin{aligned} \frac{dx}{dt} &= -ky \\ \frac{dy}{dt} &= -kx. \end{aligned}$$

   - Find the general solution to this system.
   - Assume that x_0 is the larger force. Show that the "X-men" will eventually eliminate the "Y-forces" entirely. This may seem obvious or intuitive, but you should present a coherent step-by-step argument.
   - Find the particular solution that satisfies $x(0) = x_0$, $y(0) = y_0$. (i.e. determine the constants in your general solution in terms of x_0 and y_0.)
   - Let t_v denote the moment when the "Y-forces" are vanquished. (i.e. when $y(t_v) = 0$.) Show that $e^{kt_v} = \sqrt{\frac{x_0+y_0}{x_0-y_0}}$.
   - Now show that $x(t_v) = \sqrt{x_0^2 - y_0^2}$. This is the square law of military strategy. It gives an estimate for the number of forces remaining after an encounter with another, smaller, force.

   A historically important application of the square law took place in the Battle of Trafalgar in 1805. Nelson led a British fleet of 40 ships against a French-Spanish fleet of 46 ships. (The actual number of ships has been simplified, as the ships of each fleet were of varying strength.) Nelson's battle plan, written before the battle, was to divide the French-Spanish fleet into two groups of 23. He would defeat one of these groups of 23 with 32 of his ships, while the 8 remaining ships held off the other 32. After these two skirmishes were completed, he would turn the remainder of his fleet against what was left of the Franco-Spanish fleet.

- Apply the square law to show that at the end of the day, this leaves Nelson with a force of about $5\frac{1}{2}$ ships.
- Show that in a direct encounter, Nelson would be defeated, with approximately 23 French and Spanish ships remaining.

2. In this problem, we consider a predator that hunts two (equally nutritious) prey species. For instance a (small) python hunting mice and hamsters. The model we will consider is:

$$\frac{dp}{dt} = -(0.3)p + (0.9)pm + (0.9)ph$$

$$\frac{dm}{dt} = +(0.3)m - (0.4)pm$$

$$\frac{dh}{dt} = +(0.5)h - \gamma ph$$

- What are the equilibrium points of this system? Do they depend on γ? Can the population ever stabilize when all three populations are positive?
- Consider the case where $\gamma = .6$. Look at the solution with initial conditions $p = .31$, $m = .17$ and $h = .2$. What is the fate of the pythons? (i.e. what happens to the solution as $t \to \infty$?) What is the fate of the mice and hamsters? Do both of the prey survive or do one or both die out?
- Try some different initial conditions? Does the fate of the species depend on the initial conditions?
- Use some different values of the parameter γ. How does the value of γ affect the fate of the populations? Can you explain these effects in terms of the interactions of the species?
- Suppose the mice and hamsters compete for the same resources. How would you modify the system to account for this?

3. In this problem we will explore how Pat's preference for music changes over time. We will measure Pat's preference with the quantities R and C. When R is positive, Pat loves rock music, when R is negative, Pat hates rock music. Similarly, C measures Pat's like or dislike of country music.

Here are the assumptions about Pat's musical taste:

(a) Pat will only listen to Rock music or Country music.
(b) When Pat listens to Rock music, Pat gets tired of hearing the same songs over and over. Consequently, R decreases at a rate proportional to R, and C increases at a rate proportional to R.
(c) When listening to a lot of country music, Pat begins to think it is way to twangy, but if Pat avoids the country music scene, Pat looses touch, and hence interest in country music. Consequently R increases at a rate proportional to C^2.

(d) When avoiding rock music, Pat begins to miss Pat's old favorites, but when Pat truly delves into the rock scene, Pat forgets about all the country singers. Consequently, C decreases at a rate proportional to R^2.

(e) As Pat ages, Pat's musical tastes tend to mellow out. Consequently, C tends to increase at a constant rate.

- Will Pat's musical tastes ever stabilize? If so, what will Pat's opinions be?
- What happens if Pat is initially a rock fanatic, ambivalent toward country music? A country music hater ambivalent toward rock? A hater of rock music ambivalent toward country? A hater of country ambivalent toward rock?