Models of Randomness

Part II: random types-as-closures

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Notation and concepts

SK = combinatory algebra
SKR = random extension ($R = \lambda x, y.x + y$)
  (note $(x + y) + z \neq x + (y + z)$; really $\frac{x+y}{2}$)
SKJ = parallel/ambiguous extension ($\mathbf{J} = \lambda x, y.x \mid y$)
  (some authors write join $x \mid y$ instead $x \sqcup y$)
SKJR = random+ambiguous extension

All four assume HP-complete observational equivalence:
  $M \sqsubseteq M'$ iff $\forall \underline{N}.\ M\ \underline{N}$ conv $\implies M'\ \underline{N}$ conv
where $\underline{N}$ ranges over traces $N_1, \ldots, N_k$ of arguments.

Definable types-as-closures builds on 2007 talk,
`http://www.math.cmu.edu/~fho/notes/`

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$$\text{Rand} \;:\; \text{type} \;\longrightarrow\; \text{type}$$

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\mathsf{Rand} & : & \mathsf{type} \to \mathsf{type} \\
\mathsf{always} & : & \forall\alpha.\ \alpha \to \mathsf{Rand}\ \alpha \\
\mathsf{sample} & : & \forall\alpha,\beta.\ \mathsf{Rand}\ \alpha \to (\alpha \to \mathsf{Rand}\ \beta) \to \mathsf{Rand}\ \beta
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Is Rand compatible with types-as-closures?
extend SK to random lattice SKJR.
look for "small" definable subspaces.

Abstract probability algebras

Start with a dcpo with $\perp$.

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$x + x = x$	*idempotence*
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$(w + x) + (y + z) = (w + z) + (y + x)$	*associativity*

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Compare with initial join-semilattice ("J-algebra")

$$x \mid x = x$$
$$x \mid y = y \mid x$$
$$x \mid (y \mid x) = (x \mid y) \mid z$$

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and embed SK into SKR via always $= \lambda \mathsf{x}.\mathsf{x} : \forall \alpha.\alpha \to \mathsf{Rand}\ \ \alpha$.

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and embed SK into SKR via always $= \lambda \mathsf{x}.\mathsf{x} : \forall \alpha.\alpha \to \mathsf{Rand}\ \ \alpha$.
How to sample, raising $\alpha \to \mathsf{Rand}\ \ \beta$ to $\mathsf{Rand}\ \ \alpha \to \mathsf{Rand}\ \ \beta$?

Sampling booleans

Introduce $\mathbf{K} = \lambda \mathsf{x}, \mathsf{y}.\mathsf{x}$, $\mathbf{F} = \lambda \mathsf{x}, \mathsf{y}.\mathsf{y}$ with typing

$$\frac{}{\mathbf{K} : \mathsf{bool}} \qquad \frac{}{\mathbf{F} : \mathsf{bool}} \qquad \frac{\mathsf{M} : \mathsf{bool} \qquad \mathsf{N}, \mathsf{N}' : \alpha}{\mathsf{M}\ \ \mathsf{N}\ \ \mathsf{N}' : \alpha}$$

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We can sample with

$$\mathsf{sample}_{\mathsf{bool}} = \lambda \mathsf{p}, \mathsf{f}.\ \ \mathsf{p}\ \ (\mathsf{f}\ \ \mathbf{K})\ \ (\mathsf{f}\ \ \mathbf{F})$$

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$$\mathsf{sample}_{\mathsf{bool}} = \lambda p, f.\;\; p \;\; (f \;\; \mathbf{K}) \;\; (f \;\; \mathbf{F})$$

Letting, e.g.,

$$\mathsf{amb} \;\; := \;\; \lambda x.\;\; x \;\; (x \;\; \mathbf{K} \;\; \mathbf{F}) \;\; (x \;\; \mathbf{F} \;\; \mathbf{K}) \qquad \textit{ambiguity}$$

$$R \;\; := \;\; \mathbf{K} + \mathbf{F}, \qquad \textit{even coin toss}$$

note the difference between

amb R

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$\mathsf{amb}\ \mathsf{R} \ = \ \mathbf{K} + \mathbf{F}$ *random*

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Letting, e.g.,

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note the difference between

$$\text{amb } R \ = \ \mathbf{K} + \mathbf{F}$$ *random*

$$\text{sample}_{\text{bool}}\ x \text{ from } R \text{ in amb } x$$ *...sugar for*

$$= \ \text{sample}_{\text{bool}}\ R\ \lambda x.\ \text{amb}\ x$$

$$= \ \mathbf{K}$$ *deterministic*

Sampling natural numbers

Let $\mathsf{zero} = \lambda\_, \mathsf{x.x}$, $\mathsf{succ} = \lambda \mathsf{n}, \mathsf{f}, \mathsf{x}.\ \mathsf{f}(\mathsf{n}\ \ \mathsf{f}\ \ \mathsf{x})$ with typing

$$\frac{}{\mathsf{zero} : \mathsf{nat}} \qquad \frac{}{\mathsf{succ} : \mathsf{nat} \rightarrow \mathsf{nat}} \qquad \frac{\mathsf{n} : \mathsf{nat} \qquad \mathsf{s} : \alpha \rightarrow \alpha \qquad \mathsf{z} : \alpha}{\mathsf{n}\ \ \mathsf{s}\ \ \mathsf{z} : \alpha}$$

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and sample with

$$\mathsf{sample}_{\mathsf{nat}} = \lambda \mathsf{p}, \mathsf{f}.\ \ \mathsf{p}\ \ (\mathsf{f} \circ \mathsf{succ})\ \ (\mathsf{f}\ \ \mathsf{zero})$$

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Let $\text{zero} = \lambda\_, x.x$, $\text{succ} = \lambda n, f, x.\ f(n\ f\ x)$ with typing

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and sample with

$$\text{sample}_{\text{nat}} = \lambda p, f.\ p\ \ (f \circ \text{succ})\ \ (f\ \ \text{zero})$$

then lift $\text{succ} : \text{nat} \to \text{nat}$ to $\text{succ}' : \text{Rand}\ \ \text{nat} \to \text{Rand}\ \ \text{nat}$

$$\text{succ}' = \lambda p.\ \text{sample}_{\text{nat}}\ \ p\ \ \text{succ}$$

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Moral Church terms are already monadic

Sampling partial terms

General method for defining $\mathsf{sample}_\alpha\ \mathsf{p}\ \mathsf{f}$:

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General method for defining sample_α p f:

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...but first look at typing in lattice models.

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$$\mathbf{I} \sqsubseteq \alpha = \alpha \circ \alpha$$ *extensive, idempotent*

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To minimize garbage, look for closures with small ranges.

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unit, bool, nat,

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General idea: join represents ambiguity.
When expecting a bool x, squint
(join over all ways of looking at x).
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Specifically: argue about action on Bohm trees.

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(see 2007 talk on definable closures for details)

A lattice with randomness

Seeking a random monad in a lattice model, consider the free JR-algebra over SK, where application right-distributes over both

$$(\mathrm{f} \mid \mathrm{g})\mathrm{x} = \mathrm{f}\ \ \mathrm{x} \ \ \mid \ \ \mathrm{g}\ \ \mathrm{x}$$
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Application right-distributes over slurries,
but RJ distributivity fails,
so slurries can be infinitely deep.

JR-Bohm-trees

JR-Bohm trees are defined by

$$\frac{x \ \ \text{var} \qquad M_1, \ldots, M_k \ \ \text{BT}}{x \ M_1 \ \ \ldots \ \ M_k \ \ \text{BT}} \qquad \frac{x \ \ \text{var} \qquad M \ \ \text{BT}}{\lambda x.M \ \ \text{BT}}$$

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Theorem

Every SKJR-term is equivalent to a slurry of BTs.

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Historical operational semantics for join is parallelism.

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For example...

Random convergence/divergence

The smallest nontrivial closure is

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$M \sqsubseteq M'$ iff $\forall$trace $\underline{N}$. $\text{conv}(M\ \underline{N}) \leq \text{conv}(M'\ \underline{N})$.

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(recall we want small ranges, few inhabitants)

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Question What is the spectrum of dimensions?

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Question What is the spectrum of dimensions?

Known At least $\{0, 1, 2^{\aleph_0}\}$.

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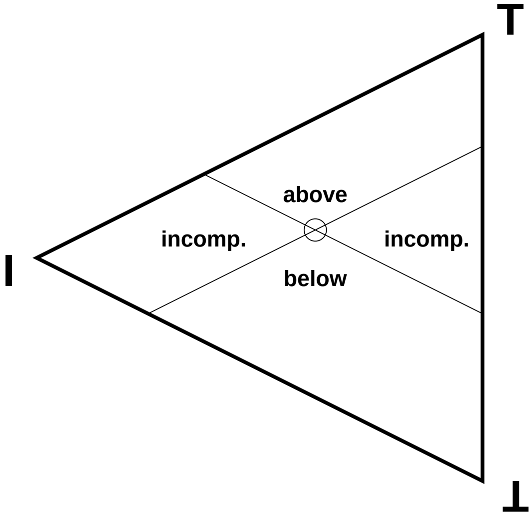
Can we disambiguate to finite dimensional mixtures, without determinizing? (unknown)

Mixtures of semibooleans

The space of $\bot, \mathbf{I}, \top$-mixtures is 2-dimensional:



Slurries of semibooleans

Theorem

$\bot, \mathbf{I}, \top$-slurries are completely characterized by their action on the unit interval $[\bot, \top]$.

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$\perp, \mathbf{I}, \top$*-slurries are completely characterized by their action on the unit interval* $[\perp, \top]$.

We thus have JR-isomorphism with the monotone convex functions $: [0, 1] \to [0, 1]$.

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This space has dimension $2^{\aleph_0}$.

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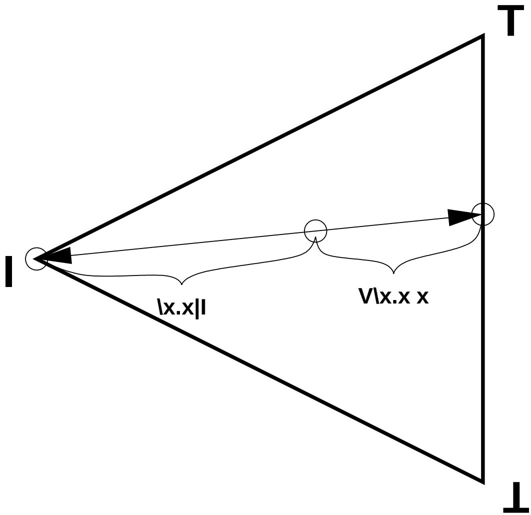
This space has dimension $2^{\aleph_0}$.
Can we raise to a finite-dimensional subspace?

A 1-dimensional subspace

Try raising with Vλx.x x and then Vλx.x | $\mathbf{I}$:



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The closure Vλx.x x is more useful
but has infinite dimensional range.

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$$\{\ (\mathsf{b}, \mathsf{k}, \mathsf{f}, \mathsf{j}, \mathsf{t}) \in [0,1]^5 \\ \mid\ \mathsf{k} \geq \mathsf{b},\ \ \mathsf{f} \geq \mathsf{b},\ \ \mathsf{j} + \mathsf{b} \geq \mathsf{k} + \mathsf{f},\ \ \mathsf{t} \geq \mathsf{j}\ \}$$

partially ordered componentwise.

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but then requires sequentialization...