Tutorial: Generic Elementary Embeddings

Lecture 3: Towers

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Corrections

Additions

A partial ordering $\mathbb{P}$ is weakly (ρ, δ)-saturated iff given any collection of antichains $\langle \mathcal{A}_\alpha : \alpha < \gamma \rangle$ where $\gamma < \rho$, there is a dense collection of $p \in \mathbb{P}$ such that for all $\alpha < \gamma$

$$|\{q \in \mathcal{A}_\alpha : p \wedge q \neq 0\}| < \delta.$$

A forcing exercise: Let $\gamma < \kappa$ be regular cardinals and $|\mathbb{P}| \leq \kappa$. Then

- $\mathbb{P}$ is weakly (γ^+, κ)-saturated iff
- after forcing with $\mathbb{P}$, $cf(\kappa) > \gamma$.

Generic Elementary embeddings

Suppose that $\mathbb{Q}$ is a partial ordering such that if $H \subset \mathbb{Q}$ is generic then there is a generic elementary embedding

$$j : V[H] \to N$$

defined over $V[H]^{\mathbb{P}}$. Then there is a generic elementary embedding

$$j' = j \restriction V : V \to M$$

defined in $V^{\mathbb{Q} * \mathbb{P}}$.

Example

If $\mathbb{Q} = Col(\omega_1, < \kappa)$, where κ is Woodin, then there is an ω_2-saturated ideal I on ω_1 in $V^{\mathbb{Q}}$.

Hence in $V^{\mathbb{Q} * P(\omega_1)/I}$, there is a generic elementary embedding $j : V \to M$ whose critical point is ω_1, sends ω_1 to κ and is such that $M^\omega \cap V^{\mathbb{Q} * P(\omega_1)/I} \subset M$.

Can we get at the embedding directly?

Projections

Let $\pi : Z \to Z'$ be a surjective function. If $I \subset P(Z)$ is an ideal then π determines an ideal I' on Z' by setting:

$$A' \in I' \text{ iff } \pi^{-1}[A] \in I.$$

I' is called a projection of I. In this case we get a natural Boolean algebra embedding

$$\iota : P(Z')/I' \to P(Z)/I.$$

Towers

Let $\langle U, <_U \rangle$ be a linearly ordered set. A collection of ideals $\langle I_u : u \in U \rangle$ is a tower if there is a commuting system $\{\pi_{u,u'} : u < u'\}$ such that I_u is the projection of $I_{u'}$ via $\pi_{u,u'}$.

Given an tower of ideals we get Boolean algebra embeddings $\iota_{u,u'} : P(Z_u)/I_u \to P(Z_{u'})/I_{u'}$.

We can take the direct limit, $\mathbb{B}_\infty$.

We will often abuse notation and view $a \in P(Z_u)/I_u$ as an *element* of $P(Z_{u'})/I_{u'}$.

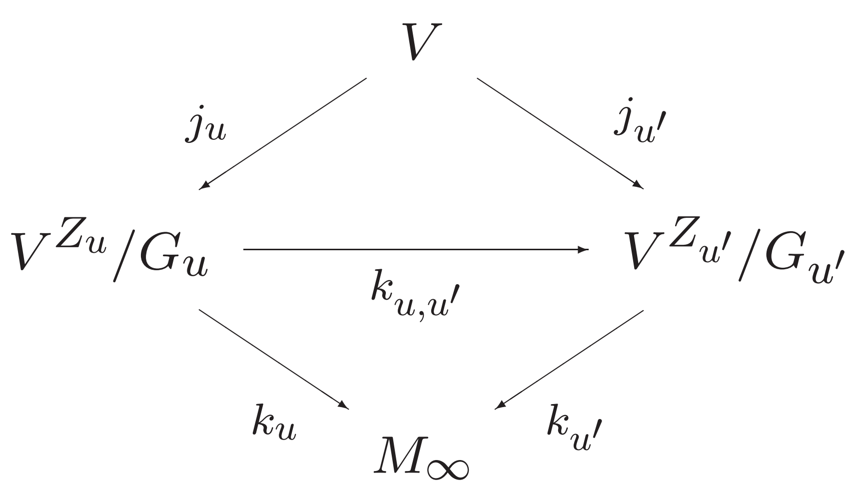
Ultrapowers by towers

Forcing with $\mathbb{B}_\infty$ yields a system of ultrafilters G_u on the Z_u's, and embeddings

$$k_{u,u'} : V^{Z_u}/G_u \to V^{Z_{u'}}/G_{u'}.$$

The direct limit of these models, M_∞ is defined to be the generic ultrapower by the tower.

So we get a commuting diagram:



Concretely

In practical situations we will have $Z \subset P(X)$, $Z' \subset P(X')$ with $X' \subseteq X$. The projection maps will be of the form:

$$\pi : P(Z) \to P(Z')$$

defined by $\pi(A) = \{z \cap X' : z \in A\}$.

Even more: we will have $X = \lambda$ and $X' = \lambda'$ and we have a tower of ideals $\langle I_\lambda : \lambda < \delta \rangle$, for some $I_\lambda \subseteq PP(\lambda))$ and for $\lambda < \lambda', z \in Z_{\lambda'}$

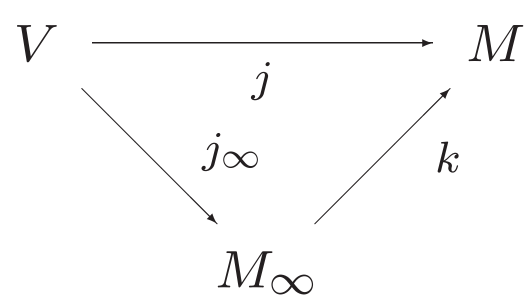
$$\pi_{\lambda,\lambda'}(z) = z \cap \lambda'.$$

In the previous example

If κ is Woodin, and $G * H \subset Col(\omega_1, < \kappa) * P(\omega_1)/I$ is generic and $j : V \to M$ is the generic embedding, then for each $\lambda < \kappa$,

$$\left\{ j\text{``}\lambda \in M \cap [j(\lambda)]^{<\omega_1} \right\} \cap V[G * H].$$

Hence we get an ideal I_λ on $P([\lambda]^{<\omega_1})$ induced by $U(j, j\text{``}\lambda)$. These form a tower and the tower forcing is a regular subalgebra of $Col(\omega_1, < \kappa) * P(\omega_1)/I$. Moreover there is a commutative diagram:



where M_∞ is the ultrapower by the tower.

Upshot

If κ is Woodin, there is an ultrapower by a tower yielding a well-founded model M_∞ and an embedding $j : V \to M_\infty$ such that:

1. crit$(j) = \omega_1$ and $j(\omega_1) = \kappa$,

2. M is closed under $<\kappa$-sequences and

3. the forcing preserves κ.

Relationship to the Kunen Construction

A very similar argument applied to the second stage of the "Kunen Construction" of a saturated ideal, yields the consistency of tower forcings that of inaccessible height δ that are δ-saturated.

General Tower Theory

Conventions: δ is a strong limit cardinal, $U \subseteq \delta$ is an unbounded set of cardinals and

$$\mathcal{T} = \langle I_\alpha \subseteq PP(H(\alpha)) : \alpha \in U \rangle$$

is a tower of normal, fine, countably complete ideals.

We will call $\delta = sup(U)$ the *height* of the tower. For $\alpha < \delta$, we let α^* be the least element of U greater than 2^{2^α}.

We will let $\mathcal{P}_\mathcal{T}$ denote the Boolean algebra $\mathbb{B}_\infty$ associated with $\mathcal{T}$.

Presaturation

We will say that a tower $\mathcal{T}$ of inaccessible height δ is presaturated iff forcing with $\mathcal{P}_{\mathcal{T}}$ preserves the statement "δ is a regular cardinal".

Presaturation is what you want

Suppose that $\mathcal{T}$ is a presaturated tower of inaccessible height δ. Then $\mathcal{T}$ is precipitous. If $G \subseteq \mathcal{P}_{\mathcal{T}}$ is generic and $j : V \to M$ is the generic ultrapower with M transitive then

$$M^{<\delta} \cap V[G] \subseteq M.$$

A typical example

Suppose that

1. $\rho \geq \omega_1$ is a successor cardinal,
2. each I_α is ρ-complete and concentrates on $[H(\alpha)]^{<\rho}$ and
3. $\mathcal{P}_\mathcal{T}$ is weakly (ρ, δ)-saturated.

Then $\mathcal{T}$ is presaturated. If $j : V \to M \subseteq V[G]$ is the elementary elementary embedding arising from a generic $G \subseteq \mathcal{P}_\mathcal{T}$ then $crit(j) = \rho$, $j(\rho) = \delta$ and $M^{<\delta} \cap V[G] \subseteq M$.

General Technique

We say that N is good for α iff $\alpha \in N$ and N is good for I_α.

If $b \in \mathcal{P}_\mathcal{T}$ we will define the support of b to be the least α such that $b \in P(H(\alpha))$.

Let $\mathcal{A}$ be a collection of subsets of $P(H(\alpha))$ for $\alpha \in U$ that form an antichain in $\mathcal{P}_\mathcal{T}$. A structure N captures $\mathcal{A}$ iff N is good for α and there is an $a \in N \cap \mathcal{A}$, with $a \subset P(H(\alpha))$ and $N \cap H(supp(a)) \in a$.

Catching antichains localizes them

Let $\mathcal{A}$ be a maximal antichain. Suppose that $\alpha < \delta$ and $[S] \in \mathcal{P}_{\mathcal{T}}$.

If $S \subseteq \{z \in H(\delta) : z$ captures $\mathcal{A}$ below $\alpha\}$, then $\{b \in \mathcal{A} : b$ is compatible with $[S]\} \subseteq \{b : supp(b) < \alpha\}$.

In particular, $\{b \in \mathcal{A} : b$ is compatible with $[S]\}$ has cardinality less than δ.

Weak saturation from catching antichains

Let $\rho \leq \delta$. Suppose that for all $\gamma < \rho$ and all sequences of maximal antichains $\langle \mathcal{A}_\alpha : \alpha < \gamma \rangle$ there is a dense set of $S \in \mathcal{P}_\mathcal{T}$ with an η between γ and δ such that if $N \in S$ and $\alpha \in \gamma \cap N$, then N captures $\mathcal{A}_\alpha$ below η. Then:

$$\mathcal{P}_\mathcal{T} \text{ is weakly } (\rho, \delta)\text{-saturated.}$$

Let $\mathcal{T}$ be a tower of height δ. Let $\mathcal{A}$ be a maximal antichain in $\mathcal{P}_{\mathcal{T}}$. Then $\mathcal{T}$ *can capture $\mathcal{A}$ at α* iff

1. $\mathcal{A} \cap \mathcal{P}_{\mathcal{T}_\alpha}$ is a maximal antichain in $\mathcal{P}_{\mathcal{T}_\alpha}$ and

2. whenever:

   (a) γ is between α and δ, $\sigma < \alpha$ and $\mathfrak{A}$ is a structure in a countable language expanding $\langle H(\gamma^*), \in, \Delta \rangle$

   there is a closed unbounded set of $N \prec \mathfrak{A}$ such that if:

   (b) N is good for γ,

   (c) $\{\mathcal{A} \cap \mathcal{P}_{\mathcal{T}_\alpha}, \mathcal{T}_\alpha\} \subseteq N$ and

   (d) $N^* \prec N$ has cardinality less than α

then there is an $N' \prec \mathfrak{A}$ such that

(A) N' is good for γ,

(B) $N' \cap H(\sigma) = N \cap H(\sigma)$,

(C) $N^* \prec N'$ and

(D) N' captures $\mathcal{A}$ below α.

Catching antichains implies precipitous

Suppose that $\mathcal{T}$ is a tower that captures antichains. Then $\mathcal{T}$ is precipitous.

A counterexample to precipitousness is given by an ω-sequence of antichains that form a tree. Catching them one by one gives a branch through the tree with non-empty intersection.

Presaturation

To show weak saturation, you typically have to capture more than ω many antichains. This requires more than antichain catching:

Example Let δ be Woodin. For regular $\alpha < \delta$ let Z_α be the collection of $N \in [H(\alpha)]^{<\omega_2}$ that are internally approachable by a sequence of length ω_1. Then $\langle NS \restriction Z_\alpha : \alpha$ is regular and $\alpha < \delta\rangle$ forms a precipitous tower. Further, if this tower is pre-saturated then $\Theta^{L(\mathbb{R})} < \omega_2$.

Presaturation

Let $\rho < \delta$ and δ inaccessible. A tower $\mathcal{T} = \langle I_\alpha : \alpha \in U \rangle$ of height δ will be called *ρ-complete* iff for all $\gamma < \rho$ and all increasing sequences $\langle \alpha_i : i < \gamma + 1 \rangle \subset U$ and all regular $\lambda >> \alpha_\gamma$ and all $u \in H(\lambda)$ if:

1. $\langle N_i : i \in \gamma \rangle$ is a sequence of elementary substructures of $\langle H(\lambda), \in, \Delta, u \rangle$ with $\{\langle \alpha_i : i < \gamma+1 \rangle, \langle I_\alpha : \alpha \in U \cap (\alpha_\gamma + 1) \rangle\} \subseteq N_j$ for all $j < \gamma$,

2. N_i good for α_γ and

3. $N_i \cap H(\alpha_i) = N_j \cap H(\alpha_i)$ for $i < j < \gamma$.

then there is an $N_\gamma \prec \langle H(\lambda), \in, \Delta, u \rangle$ with $\{\langle \alpha_i : i < \gamma + 1 \rangle, \langle I_\alpha : \alpha \in U \cap (\alpha_\gamma + 1) \rangle\} \subseteq N_\gamma$ that is good for α_γ and for all $i < \gamma$, $N_\gamma \cap H(\alpha_i) = N_{\alpha_i} \cap H(\alpha_i)$.

The payoff

Let $\mathcal{T} = \langle I_\alpha : \alpha \in U \rangle$ be a tower of normal, fine, countably complete ideals of inaccessible height δ. Suppose that:

1. $\mathcal{T}$ captures antichains and

2. $\mathcal{T}$ is $<\rho$-complete.

Then $\mathcal{T}$ is weakly (ρ, δ)-saturated.

An example

The previous theorem reduces the problem of presaturation to capturing antichains and completeness. Completeness can be difficult to verify:

Example Let δ be inaccessible. For regular $\alpha < \delta$, let $Z_\alpha = \{N \prec H(\alpha) : |N| < \omega_2$ and $N \cap \alpha$ is ω-closed$\}$.

Then $\langle NS \restriction Z_\alpha : \alpha < \delta \rangle$ is a $<\omega_2$-closed tower.

Natural towers

Let δ be an inaccessible cardinal and $U \subseteq \delta$ be a cofinal set. At tower

$$\mathcal{T} = \langle NS \restriction Z_\alpha : \alpha \in U \rangle$$

is called a stationary tower.

<u>Woodin's Towers</u>

1. When each $Z_\alpha = P(H(\alpha))$, the tower is called $\mathbb{P}_{<\delta}$ and

2. when each $Z_\alpha = [H(\alpha)]^{<\omega_1}$, the tower is called $\mathbb{Q}_{<\delta}$.

Woodin's Towers are presaturated

Theorem (Woodin) Let δ be a Woodin cardinal and $\mathcal{T}$ the stationary tower $\langle NS \restriction Z_\alpha : \alpha \in \delta\rangle$ where either:

1. for all $\alpha, Z_\alpha = P(H(\alpha))$ or

2. for all $\alpha, Z_\alpha = [H(\alpha)]^{<\kappa}$ for some regular uncountable cardinal $\kappa < \delta$.

Then $\mathcal{T}$ captures antichains.

Burke's Towers

Let δ be a supercompact cardinal and $\mathcal{T} = \langle NS \restriction Z_\alpha : \alpha \in U \rangle$ be an arbitrary stationary tower of height δ. Suppose that there is a Σ_1 formula $\phi(x, x', \vec{y})$ and $\vec{p} \in H(\delta)$ such that if we set

$$f(\alpha) = \alpha' \text{ iff } \phi^V(\alpha, \alpha', \vec{p})$$

then f bounds the map sending α to the least element of U above α.

Then $\mathcal{T}$ captures antichains.

Woodinized Supercompact cardinals

Definition: δ is a Woodinized supercompact cardinal iff for all $f : \delta \to \delta$ there is an $\alpha < \delta$ closed under f and a $j : V \to M$ with critical point α such that M is closed under $|V_{j(f)(\alpha)}|$-sequences.

These cardinals are between supercompact and huge cardinals in consistency strength.

Arbitrary Stationary tower forcing.

Suppose that δ is a Woodinized supercompact cardinal and $\mathcal{T}$ is a stationary tower of height δ. Then $\mathcal{T}$ captures antichains.

An example of Burke

Suppose that κ is supercompact and $\delta > \kappa$ is inaccessible. Then there is a tower of height δ that is not precipitous.

Examples of applications

(Woodin) Let δ be Woodin and $\mu < \delta$ be a regular uncountable cardinal.

For $\mu \leq \alpha < \delta$, let $Z_\alpha = [H(\alpha)]^{<\mu}$ and $\mathcal{T} = \langle NS \upharpoonright Z_\alpha : \alpha < \delta \rangle$.

Then forcing with $\mathcal{P}_\mathcal{T}$ yields a generic embedding $j : V \to M$ with $M^{<\delta} \subseteq M$.

If η is an ordinal less than δ and $\{z : z \cap \mu \in \mu\}$ and $\{z : cf(z \cap \eta) = \rho\}$ are in the generic object G, then the critical point of j is μ and in both $V[G]$ and M, the cofinality of η is $j(\rho)$.

Fixing $\mu = \eta = \aleph_{\omega+1}$ and $\rho = \aleph_{17}$ we see that we can force to preserve cardinals below $\aleph_\omega$ and change the cofinality of $\aleph_{\omega+1}^V$ to be $\aleph_{17}$.

A slight refinement

Let δ be a Woodin cardinal and ρ, μ, κ be regular with $\mu^+ \leq \rho < \kappa < \delta$. Let $\eta < \delta$ and

$Z_\alpha = \{z \in [H(\alpha)]^{<\kappa} : z \cap \kappa \in \kappa, z \cap \alpha$ is $<\mu^+$-closed and $cf(z \cap \eta) = \rho\}$.

Then $\mathcal{T} = \langle NS \restriction Z_\alpha : \alpha < \delta \rangle$ is a tower and if $G \subseteq \mathcal{P}_\mathcal{T}$ is generic, in $V[G]$:

- $cf(\eta) = \rho$ and
- for all ordinals ξ if $cf(\xi)^{V[G]} \leq \mu$ then $cf(\xi)^{V[G]} = cf(\xi)^V$.

Taking $\kappa = \eta = \aleph_{\omega+1}$ and $\mu = \aleph_{16}, \rho = \aleph_{17}$ we see that there are partial orderings that

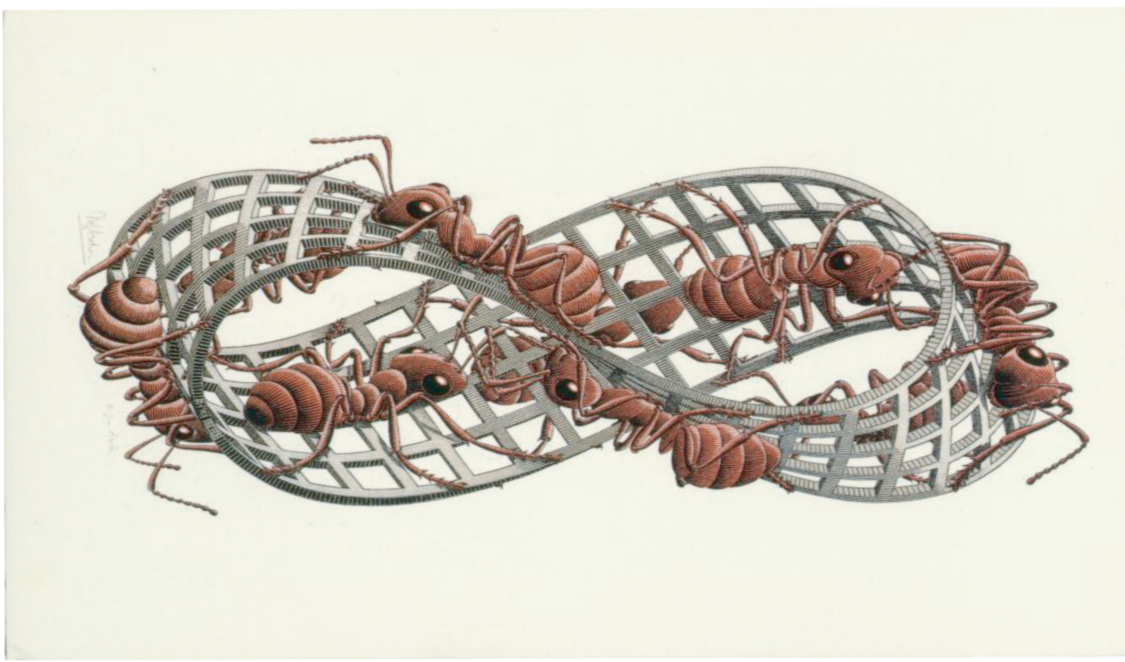
- make $\aleph_{\omega+1}$ have cofinality $\aleph_{17}$,
- preserve all cardinals below $\aleph_\omega$ and
- preserve the V-cofinality of any cardinal whose $V[G]$ cofinality is less than $\aleph_{17}$.

Another application due to Woodin

Let δ be a Woodin cardinal, $G \subset \mathbb{P}_{<\delta}$ be generic and $j : V \to M$ be the generic elementary embedding. Then in $V[G]$:

1. $j(\delta) = \delta$,
2. δ is a regular cardinal and there are unboundedly many measurable $\mu < \delta$ with $j(\mu) = \mu$ and
3. for unboundedly many measurable $\mu \in \delta$ there is a $\gamma < \mu$ and $x \subseteq \gamma$ such that $V_\mu \subseteq L[x]$.

Conventional large cardinals cannot have fixed points inside their strength.



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