Tutorial: Generic Elementary Embeddings

Lecture 2: Consistency results

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Recap

Last time:

We saw how generic elementary embedding assumptions are natural generalizations of conventional large cardinals.

We saw that they could naturally be characterized by the "three parameters".

Finally, we saw that the combinatorial device of "ideals" could be used to encode the generic elementary embedding assumptions.

Induced vs. natural ideals

The manner we got ideals from generic embeddings was to start with a generic embedding $j : V \to M \subseteq V[G]$, choose an element $i \in M$ and a Z such that $i \in j(Z)$.

This gave an ultrafilter $U(j,i) \subseteq P(Z)^V$ by setting $A \in U(j,i)$ iff $i \in j(A)$.

$U(j,i)$ in turn gave an ideal in V by setting $A \in I$ iff $\|A \in U(j,i)\| = 0$.

An ideal that arises this way is called an *induced ideal*.

Natural ideals

Natural ideals are ideal with an extrinsic definition. Examples include:

1. The non-stationary ideals
   (a) on a regular κ
   (b) on $[\lambda]^{<\kappa}$ or more generally
   (c) on $P(Z)$ for some $Z \subseteq P(\lambda)$
2. Club guessing ideals
3. $I[\lambda]$
4. Null or meager ideals
5. etc. etc.

Collapsing large cardinals

The methods for dealing with induced ideals and natural ideals differ. We begin with constructions of induced ideals with strong properties.

The basic method is to take a large cardinal embedding in a model V_0:

$$j_0 : V_0 \to M_0$$

with critical point κ and force with a partial ordering $\mathbb{P}$ to make κ an accessible cardinal such as ω_1. We then want to extend the embedding j_0 to a

$$j : V_0^{\mathbb{P}} \to M_0^{j(\mathbb{P})}.$$

Then our final model $V = V_0^{\mathbb{P}}$ will be the domain of a generic elementary embedding.

Precisely

Given $j : V \to M$ and a generic $G \subset \mathbb{P}$ we want to extend j to a

$$\hat{j} : V[G] \to M'$$

for some M'. Then we must have:

- $M' = M[H]$ for some M-generic $H \subseteq j(\mathbb{P})$
- $\hat{j}(\tau^G) = j(\tau)^H$.

Assume $\widehat{j}: V[G] \to M[H]$

Suppose that $H \subset j(\mathbb{P})$ is V-generic. Then there is an $m \in j(\mathbb{P})$ which forces that

$$j^{-1}(H) \subset \mathbb{P} \text{ is } V\text{-generic.}$$

This can be restated as:

$$j : \mathbb{P} \to j(\mathbb{P})/m \text{ is a regular embedding.}$$

Such an m is called a master condition.

More generally

We want to extend j to $\widehat{j} : V[G] \to M[H]$. It is only necessary that H be M-generic. For this it suffices to look at the filter $\mathcal{F}$ generated by $j``G$.

IF forcing with $j(\mathbb{P})/\mathcal{F}$ yields an M-generic $H \subset j(\mathbb{P})$ then we can extend the embedding.

This frequently happends in situations where you have partial master conditions.

Consistency results

Almost all consistency results for ideal axioms follow this outline. The hard work is in building the partial ordering $\mathbb{P}$ so that there is a master condition m or so that the filter $\mathcal{F}$ has nice properties.

An issue with all of the constructions is in categorizing the resulting induced ideal.

Computing the quotient of the ideal

If the original j came from the ultraproduct by an ultrafilter $U \subset P(Z)$, and $i = [id]^M$, then $\hat{j}$ yields $U(\hat{j}, i) \supseteq U$ and hence an induced ideal I.

Moreover, in most situations I is the dual of U restricted to a single set $\mathcal{M}$.

What is the partial ordering $P(Z)/I$?

Is it λ-c.c.? Is it precipitous? Presaturated? etc.

Extending generic elementary embeddings

A more general question

If the original j comes from a generic ultrapower, we force with $\mathbb{P}$ and can extend j to $\widehat{j}$, then what is the quotient forcing by the induced ideal?

Extending generic elementary embeddings

In this situation we start with a precipitous J and ask about $\bar{J}$, the ideal generated by J in the forcing extension $\mathbb{P}$.

The assertion of the existence of a master condition in this setting is:

$$id * j : P(Z)/J * \mathbb{P} \to P(Z)/J * j(\mathbb{P})/m \text{ is a regular embedding.}$$

In the special case that we are starting with a conventional ultrapower by an ultrafilter U, then $J = \bar{U}$ is prime and $P(Z)/J$ is trivial.

Duality

What we want to say is that:

$$\mathbb{P} * P(Z)/\bar{J} \cong P(Z)/J * j(\mathbb{P}).$$

This is almost true: you have to account for the master condition in $j(\mathbb{P})$.

The duality theorem

Suppose that

$$id * j : P(Z)/J * \mathbb{P} \to P(Z)/J * j(\mathbb{P})/m$$

is regular embedding.

Then there are conditions q and p so that

$$(\mathbb{P} * P(Z)/\bar{J})/q \cong (P(Z)/J * j(\mathbb{P}))/p.$$

Applications of the duality theorem

Example (classical) Suppose that κ is measurable and $\mu < \kappa$ is regular. Let $\mathbb{P} = Col(\mu, < \kappa)$ and J be the dual to a normal measure on κ.

In this case $\mathbb{P} * P(\kappa)/\bar{J} \cong P(\kappa)/J * j(\mathbb{P})$ says that in $V^{\mathbb{P}}$:

$$P(\kappa)/\bar{J} \cong j(\mathbb{P})/\mathbb{P}.$$

So:

$$P(\kappa)/\bar{J} \cong Col(\mu, < j(\kappa)).$$

In particular $P(\kappa)/\bar{J}$ contains a dense μ-closed subset.

Another example

Suppose that I is a countably complete ideal on ω_1 such that $P(\omega_1)/I \cong Col(\omega, \omega_1)$. What is the quotient algebra $P(\omega_1)/\bar{I}$ after adding ω_1-Cohen reals?

Answer

By the Duality Theorem:

$$Add(\omega, \omega_1) * P(\omega_1)/\bar{I} \cong P(\omega_1)/I * Add(\omega, \omega_2).$$

In particular in the model after adding ω_1-Cohen reals

$P(\omega_1)/\bar{I} \cong Col(\omega, \omega_1) * Add(\omega, \omega_2).$

Preservation Theorems

(Baumgartner-Taylor/ Laver) Suppose that I is a κ^+-saturated ideal on κ. Let $\mathbb{P}$ be c.c.c. Then if $G \subset \mathbb{P}$ is generic, $\bar{I}$ is κ^+-saturated in $V[G]$ iff $j(\mathbb{P})$ is κ^+-c.c. in $V[G, H]$.

Proof:

$$\mathbb{P} * P(\kappa)/I \cong P(\kappa)/I * j(\mathbb{P}).$$

More corollaries

Kakuda's Theorem Suppose that I is a precipitous ideal on κ and $\mathbb{P}$ is κ-c.c. Then $\bar{I}$ is precipitous in $V^{\mathbb{P}}$.

Theorem Suppose that I is a normal, fine, precipitous ideal on $[\lambda]^{<\omega_1}$ for some λ, and $\mathbb{P}$ is a proper partial ordering with $2^{\mathbb{P}} \leq \lambda$. Then there is a dense collection of sets $A \in P([\lambda]^{<\omega_1})/I$ such that $I \restriction A$ precipitous in $V^{\mathbb{P}}$.

In particular, if you collapse a supercompact to be ω_1, then proper forcing preserves the axiom that ω_1 is "generically supercompact."

Generalizations

There is a version of the Duality Theorem for the more complicated case where you force with $j(\mathbb{P})/\mathcal{F}$, rather than below a master condition. This allows one to compute quotient algebras in constructions that only have partial master conditions.

The Duality Theorem and its generalizations allow one to compute the quotient forcing in all cases I am aware of.

Consistency results for natural ideals

There are two different approaches for making natural ideals have nice properties.

1. Take an existing induced ideal and make it natural by changing the properties of the sets in the ideal.

2. Start with a natural ideal and manipulate its antichain structure.

Making induced ideals natural

Start with a generic elementary embedding $j_0 : V_0 \to M_0$ and let I be the induced ideal from $U(j_0, i)$ for some i.

A forcing construction is carried out that makes elements of $\breve{I}$ belong to the dual of the natural ideal. Care must be taken to make sure that the generic embedding j can be extended during the forcing construction.

The ability to extend the embedding j implies that the critical point of j remains a regular cardinal after the forcing.

A complication to this outline is that if j_α is the generic embedding after α stages of the iteration, then the induced ideal I_α for $U(j_\alpha, i)$ may properly contain the original ideal I.

Thus the construction involves:

1. A "nice" original ideal $I = I_0$.

2. An iteration $\langle(\mathbb{R}_\alpha, \dot{\mathbb{Q}}_\alpha) : \alpha < \lambda\rangle$ such that in $V_0^{\mathbb{R}_\alpha}$, the original embedding j can be generically extended to an embedding j_α from $V_0^{\mathbb{R}_\alpha}$ to $M_0^{j(\mathbb{R}_\alpha)/m_\alpha}$. Moreover, for $\alpha < \beta$ we have $m_\beta \leq m_\alpha$ and $j_\alpha \subseteq j_\beta$.

3. An sequence of ideals $\langle I_\alpha : \alpha < \lambda\rangle$ where I_α is the induced ideal for $U(j_\alpha, i)$. Since the j_α's cohere and the m_α's get stronger with α, we have $I_\alpha \subseteq I_\beta$ for $\alpha < \beta$.

4. For every element S that belongs to some I_α for an $\alpha < \lambda$ there is a β so that $\mathbb{Q}_\beta$ puts S into the natural ideal.

5. $I_\infty =_{def} \bigcup_{\alpha<\lambda} I_\alpha$ is the natural ideal in $V_0^{\mathbb{R}_\lambda}$.

The final model will be $V_0^{\mathbb{R}_\lambda}$. Typically

$$j_\infty =_{def} \varinjlim j_\alpha$$

gives a generic embedding from $V_0^{\mathbb{R}_\lambda}$ to $M_0^{j(\mathbb{R}_\lambda)/\mathcal{F}}$ and the induced ideal from $U(j_\infty, i)$ is I_∞. By property 4.), I_∞ is the natural ideal.

If the original ideal I has nice properties, e.g. that I is saturated or that I is precipitous and the embedding j was the generic ultraproduct of I, then I_∞ can be shown to retain some of these properties.

Examples of theorems like this

Theorem(Komjath) Suppose there is a measurable cardinal. Then there is a forcing extension in which there is a non-meager set $A \subseteq \mathbb{R}$ such that $P(A)/\{meager\ sets\}$ is precipitous.

Theorem(Magidor) Suppose that there is a measurable cardinal, then there is a forcing extension in which NS_{ω_1} is precipitous.

Theorem(Foreman-Komjath) Suppose there is an almost huge cardinal κ and $\mu < \kappa$. Then there is a forcing extension in which a club guessing ideal on μ^+ is saturated.

etc.

Manipulating the antichain structure

Here the ideas are *catching an antichain*, *goodness and self-genericity*.

We begin with some definitions: Let I be a normal, fine ideal on $P(Z)$ for some $Z \subset P(X)$. Let $\theta >> |Z|$.

1. $N \prec H(\theta)$ is *good* iff $N \cap X \in C$ for all $C \in N \cap \breve{I}$

2. N *catches* an antichain $\mathcal{A}$ iff N is good and there is an $a \in N \cap \mathcal{A}$ such that $N \cap X \in a$.

3. N is *self-generic* iff N catches every antichain $\mathcal{A} \in N$.

Self-genericity

The following are equivalent:

1. N is self-generic

2. $U(id, N \cap X)$ is generic over N for the forcing $P(Z)/I$.

Self-genericity and Saturation

If $|Z| = |X|$, then the following are equivalent:

- I is $|X|^+$-saturated
- Almost every good N is self-generic.

Antichain catching and precipitousness

Suppose that for all $S \in I^+$, and all sequences $\langle \mathcal{A}_n : n \in \omega \rangle$ of maximal antichains below $[S]_I$ there is an elementary substructure $N \prec H(\theta)$ with $I, Z, X, \langle \mathcal{A}_n : n \in \omega \rangle, S \in N$ such that:

1. $N \cap X \in S$ and

2. for all n, N catches an index for the antichain $\mathcal{A}_n$.

Then I is precipitous.

How to catch antichains

Basic Proposition

Suppose that κ is a Woodin cardinal and $\mu < \kappa$ is regular. Let $G \subset Col(\mu, < \kappa)$ be generic. In $V[G]$:

If $N \prec H(\theta)$, $\mathcal{A}$ is a maximal antichain in $P(\mu)/NS$, and $\mathcal{A} \in N$, then there is an N' such that:

1. $N \prec N'$
2. N' catches $\mathcal{A}$
3. $N' \cap \mu = N \cap \mu$.

Corollary

Suppose that κ is Woodin in V, and $G \subseteq Col(\mu, < \kappa)$ is generic. Then:

1. NS_μ is precipitous

2. If $(\mu, \lambda) \twoheadrightarrow (\kappa, \rho)$ then there is a precipitous ideal on $[\mu]^\kappa$ concentrating on $[\lambda]^\rho$

3. If μ is Jonsson in V then there is a precipitous ideal on $[\mu]^\mu$

4. If λ and $\mu = \lambda^+$, then there is a precipitous ideal on

$$\{N \subset \mu : |N \cap \lambda| = \lambda\}.$$

These are some of the hypothesis used to show strong results in the first lecture.

The non-stationary ideal on ω_1

Using the Basic Proposition and reflection one can show that if κ is supercompact and $G \subset Col(\omega_1, < \kappa)$, then in $V[G]$:

The non-stationary ideal on ω_1 is presaturated.

The non-stationary ideal on ω_1

We want every maximal antichain in $P(\omega_1)/NS_{\omega_1}$ to be of size $\omega_!$.

Suppose that $\mathcal{A} = \langle A_\alpha : \alpha < \omega_2 \rangle$ is a maximal antichain in $P(\omega_1)/NS$. Two tries:

1. by forcing with $Col(\omega_1, \omega_2)$

   Problem Then $\mathcal{A}$ stops being maximal.

2. by shooting a closed unbounded set through $\nabla_{\beta<\alpha} A_\beta$ for some $\alpha < \omega_2$.

   Problem This kills stationary sets and so can't be iterated.

Solution

Force with $Col(\omega_1, \omega_2) * \mathbb{Q}$ where $\mathbb{Q}$ shoots a closed unbounded set through $\triangledown\langle A_\alpha : \alpha < \omega_2^V \rangle$.

This partial ordering preserves stationary sets. Moreover, it is semi-proper iff for almost all $N \prec H(\lambda)$, there is an N' such that:

1. $N \prec N'$
2. N' catches $\mathcal{A}$
3. $N' \cap \omega_1 = N \cap \omega_1$.

Assuming sufficient large cardinals

The forcing for sealing antichains can be iterated in such a way that the non-stationary ideal on ω_2 is saturated.

This gives the consistency of the statement:

The non-stationary ideal on ω_1 is ω_2 saturated.

Another example of this technique

Ishiu showed from a Woodin cardinal that it is consistent that:

There is a club guessing sequences such that the club guessing ideal on ω_1 is ω_2-saturated.

The End