Tutorial: Generic Elementary Embeddings

Lecture 1

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These lectures serve as a very brief introduction to the article

Ideals and generic elementary embeddings,

to appear in the **Handbook of Set Theory**.

Plan of Tutorial

Lecture 1: Introduction and consequences of generic elementary embeddings.

Lecture 2: Consistency results.

Lecture 3: Forcing with towers.

Standard Large Cardinal Assumption

There is an elementary embedding

$$j : V \to M$$

where M is well founded.

Two parameters determine the strength of the embedding:

1. Where j moves ordinals

2. The closure properties of M

Variations on this can include such statements as: for all $f : \kappa \to \kappa$ there is a j with critical point $\alpha < \kappa$ such that α is closed under f and $V_{j(f)(\alpha)} \subset M$.

Generalized Large Cardinals.

Idea:

Allow j to be defined in a generic extension of V.

A typical statement is:

There is partial ordering $\mathbb{P}$ such that if $G \subset \mathbb{P}$ is generic then in $V[G]$ there is an elementary embedding

$$j : V \longrightarrow M$$

where M is well-founded.

The Three Parameters

Now three parameters determine the strength of the embedding:

1. Where j moves ordinals

2. The closure properties of M

3. The nature of the forcing $\mathbb{P}$.

For this reason we often speak of "generically huge", or "generically supercompact" cardinals.

Ideals.

Generalized large cardinals are usually discussed in terms of ideals. The translation goes as follows:

Suppose that $j : M \to N$ is an elementary embedding and $i \in j(Z)$ for some set $Z \in M$. Then there is a natural ultrafilter $U(j,i)$ associated with the *ideal element* i.

Define $U(j,i)$ on $P(Z)^M$ by setting

$$A \in U(j,i) \text{ iff } i \in j(A).$$

Then $U(j,i)$ is M-complete for intersections of size less than $crit(j)$.

Ideals Continued.

Suppose now that $\mathbb{P}$ is a partial ordering and $\dot{U}$ is a term in $V^{\mathbb{P}}$ for any ultrafilter on $P(Z)^V$. Then we get an ideal by setting:

$$A \in I \text{ iff } \|A \in \dot{U}\|^{\mathbb{P}} = 0.$$

Then I is an ideal and if U is κ-complete for sequences that lie in V, then I is a κ-complete ideal.

Putting this together

Suppose that $j : V \to M \subset V[G]$ is definable in $V[G]$ for some generic $G \subset \mathbb{P}$, $i \in M$ and $Z \in V$ is such that $i \in j(Z)$. Then in $V[G]$ there is an ultrafilter $U(j,i) \subset P(Z)^V$. This ultrafilter gives an ideal I on Z that lies in V.

We can also go the other direction–frequently ideals yield generic elementary embeddings.

Generic ultraproducts.

Suppose that I is an ideal on $P(Z)$. Then $P(Z)/I$ is a Boolean algebra. If we force with $P(Z)/I$ (without the zero element) then we get a V-ultrafilter $G \subseteq P(Z)$.

With this ultrafilter we can take the ultraproduct V^Z/G using functions in V. This gives us a generic elementary embedding

$$j : V \to V^Z/G.$$

An ideal I is precipitous if this generic ultrapower is always well-founded.

Some conventions.

- X will be a set: typically a cardinal λ
- Z will be a subset of $P(X)$: typically λ itself or $[\lambda]^{<\kappa}$.
- $I \subset P(Z)$ will be a proper ideal.
- For $A \subset Z$ not in I we write $A \in I^+$.
- $\breve{I} =_{def} \{Z \setminus A : A \in I\}$ is the dual filter to I.
- Unless otherwise stated, all of our ideals will be countably complete and proper.

Some structural properties of an ideal.

If our ideal element i is of the form $j``X$, then I has two additional properties, I is normal and fine.

Normal:

For all $A \in I^+$ and all functions $f : A \to X$ such that $f(z) \in z$ for all $z \in A$, there is a $y \in X$ such that $\{z \in A : f(z) = y\} \in I^+$.

Fine:

For all $x \in X$ the set $\hat{x} = \{z \in Z : x \in z\} \in \breve{I}$.

Standing assumption.

From now on all of our ideals will be normal, fine and countably complete.

Saturation

There are many combinatorial properties of ideals that allow one to computing the closure of the generic ultraproduct. The most fundamental is saturation.

An ideal $I \subset P(Z)$ is κ-saturated iff $P(Z)/I$ has the κ-c.c.

Local variations of this idea include presaturation, weak (κ, λ)-saturation etc.

Consequences of generic large cardinals

Strong reflection properties are the easiest fruit to pick. We will state the results in term of embeddings, rather than in terms of ideals. Assume that $j : V \to M \subset V[G]$ is a generic elementary embedding and $k, m, n \in \omega$.

- If $n < m$, $j(\omega_n) = \omega_m^V$,$j(\omega_{n+k}) = \omega_{m+k}^V$, and $j``\omega_{m+k} \in M$, then

$$(\omega_{m+k}, \omega_m) \twoheadrightarrow (\omega_{n+k}, \omega_n).$$

- Setting $k = 1, m = n + 1$ we see that the existence of such a j implies that

  square fails and there are no Kurepa trees.

- Setting $k = 2, m = n + 1$ we see that

  $2^{\omega_n} = \omega_{n+1}$ implies $2^{\omega_m} = \omega_{m+1}$.

etc.

Consequences cont.

Suppose that $j(\aleph_\omega) = \aleph_\omega$, crit$(j) < \aleph_\omega$ and $j\,``\aleph_\omega \in M$, then $\aleph_\omega$ is Jonsson.

(Much weaker ideal assumptions also yield the same conclusion.)

The CH.

Suppose that there is an elementary embedding $j : V \to M \subseteq V[G]$ with critical point ω_2 where $G \subset \mathbb{P}$ is generic for some ω-closed partial ordering $\mathbb{P}$. Then the CH holds.

Suslin Trees

Suppose that there is a $j : V \to M \subset V[G]$ such that:

1. $\mathrm{crit}(j) = \omega_1$,

2. M is closed under ω_2^V sequences in $V[G]$ and

3. $G \subseteq \mathbb{P}$ is generic where $\mathbb{P} = Col(\omega, \omega_1)$.

Then there is a Suslin tree on ω_1.

I am giving the easier versions of these results. In the cases of the Suslin tree and the CH, **Woodin** has improved these results by giving deeper arguments for theorems with different hypothesis.

State of the art

For the CH, GCH, properties of trees, descriptive set theory and reflection properties of stationary sets, generic large cardinals give a unified picture settling most independent questions.

There is one rather obscure counterexample, which we describe next.

Disunity

Let ϕ be the partition relation:

$$\binom{\omega_2}{\omega_1} \to \binom{\omega_1}{\omega_1}_\omega$$

- If there is an $(\omega_2, \omega_2, \omega)$-saturated ideal on ω_1 and the CH holds then ϕ holds.

- If there is an inaccessible λ and a normal fine ideal I on $Z = [\lambda]^{\omega_1}$ such that $P(Z)/I \cong \mathbb{B}(Col(\omega, < \lambda))$, then ϕ fails.

This is the **ONLY** known example of this type for ideal assumptions known to be consistent.

An axiom of a different form

There is an ideal axiom that is not stated in terms of the three parameters, whose consequences are radically different. This axiom is:

The non-stationary ideal on ω_1 is ω_2-saturated.

The non-stationary ideal on ω_1

Theorem (Shelah) Suppose that

$$P(\omega_1)/NS \cong Col(\omega, \omega_1).$$

Then weak diamond fails. In particular $2^\omega = 2^{\omega_1}$.

Theorem (Woodin) Suppose that the non-stationary ideal on ω_1 is ω_2-saturated and there is a measurable cardinal. Then:

$$\delta^1_2 = \omega_2.$$

Some puzzles

Is there analogue of the axiom NS_{ω_1} is ω_2-saturated for larger cardinals? Simply changing ω_1 to ω_2 doesn't work. For example, assuming that there is a measurable cardinal, the assertion that

$$NS_{\omega_2} \restriction cof(\omega_1) \text{ is } \omega_3\text{-saturated}$$

is inconsistent with the statement

$$NS_{\omega_1} \text{ is } \omega_2\text{-saturated.}$$

Another puzzle

Why are mutually contradictory "3-parameter" axioms so sparse? Is there some mutually consistent central theory?

The End