SET MAPPING REFLECTION

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1. Introduction

The goal of these lectures is to give an exposition of the concept of an *open stationary set*, an associated reflection principle (for lack of a better word), and a list of examples of how this sort of consideration arises naturally in the context of modern set theory. We will begin with a list of seemingly unrelated questions.

Question 1.1. Does PFA imply there is a well ordering of $\mathcal{P}(\omega_1)$ which is definable over $\langle H(\aleph_2), \in\rangle$ (with parameters)?

Question 1.2. Is it consistent that every Aronszajn line contains a Countryman suborder?

Question 1.3. Is it consistent that for all $c\colon [\omega_1]^2 \to 2$ there exist $A, B \in [\omega_1]^{\omega_1}$ such that c is constant on $\{\{\alpha, \beta\} : \alpha < \beta \wedge \alpha \in A \wedge \beta \in B\}$?

Let us focus on the second question for a moment. Consider the following analogy. Recall that a forcing Q satisfies the *countable chain condition (c.c.c.)* if every uncountable collection of conditions in Q contains two compatible conditions. Similarly, a forcing Q satisfies *Knaster's Condition (Property K)* if every uncountable collection of conditions contains an uncountable subcollection of pairwise compatible conditions. It is easily verified that the product of a c.c.c. forcing and one with Property K is c.c.c.. A consequence of this is that a Property K forcing cannot destroy a counterexample to Souslin's Hypothesis. Hence while the forcing axiom for c.c.c. forcings (a.k.a. $\mathrm{MA}_{\aleph_1}$) does imply Souslin's Hypothesis, the forcing axiom for Property K forcings is consistent with the failure of Souslin's Hypothesis.

What if the common and widely successful methods for building proper forcings inadvertently satisfied a stronger form of properness and that counterexamples to Question 1.2 were preserved by this stronger condition?

It turns out that this is indeed the case and we will now formulate a combinatorial obstruction of this sort. A $\mho$-*sequence* is a sequence $\langle f_\alpha : \alpha < \omega_1\rangle$ of continuous functions $f_\alpha : \alpha \to \omega$ such that if $E \subseteq \omega_1$ is closed and unbounded, there is a δ such that f_δ takes all values in ω on $E \cap \delta$. Notice that if $f : \delta \to \omega$ is continuous and δ is a limit ordinal, then there is a cofinal $C \subseteq \delta$ of ordertype ω such that $f(\xi)$ depends only on $|C \cap \xi|$. That is f is obtained by coloring the intervals in δ between points of C. Jensen's principle $\diamondsuit$ easily implies the existence of a $\mho$-sequence. Since only the club filter is quantified over in the definition of a $\mho$-sequence, $\mho$-sequences are preserved by c.c.c. forcing. This is because if E is a club in a c.c.c. forcing extension, E contains a club from the ground model (this appears as an exercise in [9]). In fact a much broader class of proper forcings preserve $\mho$-sequences; this will be discussed more later. In [12] it was shown that the existence of a ℧-sequence implies the existence of an Aronszajn line with no Countryman suborder.

While Question 1.3 has a negative answer [16], the construction in [12] served as a precursor to the ZFC construction of a coloring c as in Question 1.3 (even though [12] was published considerably after [16]). We will see that a positive answer to Question 1.1 holds and that this is related to the existence of a weak analogue of a ℧-sequence which exists on $[\omega_2]^\omega$.

The focus of this note will be to examine a principle, MRP, which provides a general framework for eliminating combinatorial obstructions such as ℧-sequences and for tapping into additional strength of the Proper Forcing Axiom (PFA). After defining the principle, we will present a number of case studies of how this principle is applied.

The reader is assumed to have familiarity with set theory at the level of Kunen's [9]. Additional background can be found in [7]. In order to make the discussion of consistency less cumbersome, we will generally assume unless otherwise stated that the existence of a supercompact cardinal is consistent.

2. The club filter and stationary sets

Central to our discussion will be the "club filter" of countable sets on a given uncountable set X. Henceforth, our convention is that X is an uncountable set, θ is an uncountable regular cardinal, and $[X]^\omega = \{A \subseteq X : |A| = \aleph_0\}$.

Definition 2.1. The *Ellentuck topology* on $[X]^\omega$ is generated by the basic open sets

$$[a, N] = \{A \in [X]^\omega : a \subseteq A \subseteq N\}$$

where a ranges over $[X]^{<\omega}$ and N ranges over $[X]^\omega$.

It is not difficult to show that in fact the basic open sets in this topology are closed as well and hence that the topology is regular and Hausdorff.

Definition 2.2. A *club*[1] in $[X]^\omega$ is a subset that is Ellentuck closed and cofinal in $([X]^\omega, \subseteq)$.

Observe that if $X = \omega_1$, then ω_1 is club when viewed as a subset of $[\omega_1]^\omega$. Hence every closed unbounded subset of ω_1 is club when viewed as a subset of $[\omega_1]^\omega$ and if $E \subseteq [\omega_1]^\omega$ is club, then $E \cap \omega_1$ is closed and unbounded.

The two other competing definitions of "club" which occur in the literature are (i) sets of the form E_f and (ii) subsets E of $[X]^\omega$ which are cofinal and closed under unions of countable chains. The following facts show that this is an intermediate notion. In particular, the definition of *stationary* does not depend on which definition is used.

Definition 2.3. $S \subseteq [X]^\omega$ is *stationary* if $S \cap E \neq \emptyset$ for every club E.

Fact 2.4. *If $f\colon X^{<\omega} \to X$ and $E_f = \{M \in [X]^\omega : f``M^{<\omega} \subseteq M\}$, then E_f is club. Moreover if E is club, then there is a $f\colon X^{<\omega} \to X$ such that $E_f \subseteq E$.*

If f is as in the above definition and $f''M^{<\omega} \subseteq M$, then we say that M is *closed under f*.

[1] Note that "club" is a misnomer since it suggests the meaning of being "closed and unbounded." In fact it means closed and cofinal.

Fact 2.5. *If $E \subseteq [X]^\omega$ is club and $\mathcal{N} \subseteq E$ is countable and linearly ordered by $\subseteq$, then $\cup\mathcal{N}$ is in E.*

The next fact states the quintessential properties of clubs and stationary sets.

Fact 2.6. *A countable intersection of clubs is a club. Equivalently, a partition of a stationary set into countably many pieces has a stationary piece.*

3. Elementary submodels

Unless otherwise specified, θ will denote a regular uncountable cardinal.

Definition 3.1. $H(\theta)$ is the collection of all sets of hereditary cardinality less than θ. We will identify $H(\theta)$ with the structure $(H(\theta), \in)$.

The following observations are useful. Some require proof (which we leave to the reader).

(1) $H(\theta)$ is a set (not a proper class) of cardinality $2^{<\theta}$.
(2) $\langle H(\theta), \in\rangle$ satisfies ZFC except possibly the power set axiom.
(3) $\mathrm{O}rd^{H(\theta)} = \theta$.
(4) If $A, B \in H(\theta)$, then $A \times B \in H(\theta)$.
(5) If $A \in H(\theta)$, then $\mathcal{P}(A) \subseteq H(\theta)$. In particular, if $A, B \in H(\theta)$, then $|A| = |B|$ if and only if $H(\theta) \models |A| = |B|$.
(6) $H(\theta^+)$ is an element of $H(2^{\theta^+})$.

Definition 3.2. We say M is a *countable elementary submodel* of $H(\theta)$ and write $M \prec H(\theta)$ if $M \in [H(\theta)]^\omega$ and, for every logical formula φ with parameters in M, $M \models \varphi$ if and only if $H(\theta) \models \varphi$.

Note our convention that $M \prec H(\theta)$ always implies $|M| = \aleph_0$. This is not standard, but it will considerably simplify writing at times.

Fact 3.3. *There is a function $f\colon H(\theta)^{<\omega} \to H(\theta)$ such that if $M \in [H(\theta)]^\omega$ and M is closed under f, then $M \prec H(\theta)$.*

Fact 3.4. *If $M \prec H(\theta)$ and $X \in H(\theta)$ and X is definable from parameters in M, then $X \in M$.*

Fact 3.5. *If $M \prec H(\theta)$, then $M \cap \omega_1$ is a countable ordinal that is not in M.*

Fact 3.6. *If $X \in H(\theta)$ is uncountable and $A \in [H(\theta)]^{\leq\omega}$, then $\{M \cap X : A \subseteq M \prec H(\theta)\}$ contains a club.*

Fact 3.7. *If $A \in M \prec H(\theta)$ and $A \not\subseteq M$, then A is uncountable. Also, if $M \prec H(\theta)$, then, for all $A \in M \cap \mathcal{P}(\omega_1)$, A is uncountable if and only if $A \cap M$ is unbounded in $\omega_1 \cap M$.*

Fact 3.8. *If $X, S \in M \prec H(\theta)$ and $S \subseteq [X]^\omega$ and $M \cap X \in S$, then S is stationary.*

Fact 3.9. *$\{M : M \prec H(\theta^+)\}$ is in $H(2^{\theta^+})$ but is not definable in $H(\theta^+)$.*

For more of the basics of Stationary sets, see Chapter 8 of Jech [7]. From this point on it will be convenient to adopt the convention that, unless otherwise stated, X is an uncountable set which is an element of $H(\theta)$.

4. The strong reflection principle

Before formulating MRP, we will first define the simpler *Strong Reflection Principle (*SRP*)* of Todorcevic [1]. We will then recall some conclusions and arguments which will serve as a starting point for the development of MRP. The material in this section is based on [1].

Recall that if $M, N \prec H(\theta)$ and λ is a cardinal, we say that N λ*-end extends* M if $M \cap \lambda = N \cap \lambda$ and $M \subseteq N$. We will only be interested in ω_1-end extensions and will refer to them simply as *end extensions.* The following fact will be used frequently.

Fact 4.1. *If $\langle N_\xi : \xi < \omega_1 \rangle$ is a continuous $\in$-increasing sequence of countable elementary submodels of some $H(\theta)$, and $\overline{N}$ is a countable elementary submodel of $H(\theta)$ with $\langle N_\xi : \xi < \omega_1 \rangle$ in $\overline{N}$, then $\overline{N}$ end extends N_δ where $\delta = \overline{N} \cap \omega_1$.*

Proof. By Fact 3.7, N_ξ is a subset of $\overline{N}$ for all $\xi < \delta$. By continuity of $\langle N_\xi : \xi < \omega_1 \rangle$, $N_\delta \subseteq \overline{N}$. Also, by continuity of $\langle N_\xi : \xi < \omega_1 \rangle$, the map $\xi \mapsto N_\xi \cap \omega_1$ is continuous. It follows from Fact 2.4 that $E = \{\xi < \omega_1 : N_\xi \cap \omega_1 = \xi\}$ is a club. Since this club is in $\overline{N}$, it contains δ as an element by Fact 3.8 and hence $N_\delta \cap \omega_1 = \delta = \overline{N} \cap \omega_1$. $\square$

SRP asserts that if $X \in H(\theta)$ with X uncountable and $S \subseteq [X]^\omega$, then there exists a continuous $\in$-chain $\langle N_\xi : \xi < \omega_1 \rangle$ of elementary submodels of $H(\theta)$ such that, for all $\xi < \omega_1$, $X \in N_\xi$ and we have $N_\xi \cap X \in S$ if and only if there exists an end extension M of N_ξ such that $M \cap X \in S$. We say such an $\langle N_\xi : \xi < \omega_1 \rangle$ is a *strong reflecting sequence* of S. The power of the continuity assumption lies in the ability to generate end extensions via Fact 4.1.

Recall that if $S \subseteq [X]^\omega$ is stationary, then we say S *reflects* if there is a continuous $\in$-chain $\langle N_\xi : \xi < \omega_1 \rangle$ of countable elementary submodels of $H(\theta)$ such that $\{\xi < \omega_1 : N_\xi \cap X \in S\}$ is stationary. The following proposition justifies the "strong" in Strong Reflection Principle.

Proposition 4.2. *If $S \subseteq [X]^\omega$ is stationary and $\langle N_\xi : \xi < \omega_1 \rangle$ strongly reflects S, then $\Xi = \{\xi < \omega_1 : N_\xi \cap X \in S\}$ is stationary.*

Proof. Suppose not and let $E \subseteq \omega_1$ be a club disjoint from Ξ. Choose $M \prec H(\theta)$ such that $E, \langle N_\xi : \xi < \omega_1 \rangle \in M$ and $M \cap X \in S$. By Fact 4.1, M is an end extension of N_δ where $\delta = M \cap \omega_1$. Notice also that δ is in E by Fact 3.8 and hence N_δ is not in S. But this is a contradiction to our assumption that $\langle N_\xi : \xi < \omega_1 \rangle$ is a strong reflecting sequence for S. $\square$

Proposition 4.3. SRP *implies that if S_ξ $(\xi < \omega_2)$ are stationary subsets of ω_1, then there are $\xi < \eta$ such that $S_\xi \cap S_\eta$ is stationary.*

Proof. Let $\langle S_\xi : \xi < \omega_2 \rangle$ be given. Define $\Gamma \subseteq [\omega_2]^\omega$ to be the collection of all P such that $P \cap \omega_1$ is an ordinal and there is an α in P such that $P \cap \omega_1$ is in S_α.

Applying SRP, there is a continuous chain N_ξ $(\xi < \omega_1)$ of countable elementary submodels of $H(\aleph_3)$, each containing $\langle S_\xi : \xi < \omega_2 \rangle$ as a member, and such that for all $\xi < \omega_1$, if N_ξ has an end extension $\bar{N}$ with $\overline{N} \cap \omega_2$ in Γ, then $N_\xi \cap \omega_2$ is in Γ. Since $\bigcup_{\xi < \omega_1} N_\xi$ has cardinality ω_1, it suffices to show that if $\beta < \omega_2$, then there is an α in some $N_\xi \cap \omega_2$ such that $S_\alpha \cap S_\beta$ is stationary. To this end, let β be given and let $\overline{N}$ be a countable elementary submodel of $H(\aleph_3)$ such that $\langle N_\xi : \xi < \omega_1 \rangle$ and β are in $\overline{N}$ and $\delta = \overline{N} \cap \omega_1$ is in S_β. By Fact 4.1, $\overline{N}$ is an end extension of N_δ

which is moreover in Γ. By assumption, N_δ is in Γ and therefore there is an α in N_δ such that δ is in S_α. Finally, by Fact 3.8, $S_\alpha \cap S_\beta$ is stationary since it contains $\delta = \overline{N} \cap \omega_1$. $\square$

5. The Set Mapping Reflection Principle

Now we will turn to the Set Mapping Reflection Principle (MRP).

Definition 5.1. Let X, θ be fixed. Suppose Σ is a map such that $\mathrm{dom}(\Sigma)$ is a club subset of $\{M : M \prec H(\theta)\}$, and $\Sigma(M) \subseteq [X]^\omega$ for all M. We say Σ is an *open set mapping* if $\Sigma(M)$ is open (in the Ellentuck topology) for all M.

Typically, $\Sigma(M)$ will actually be a subset of $[M \cap X]^\omega$.

Definition 5.2. We say $S \subseteq [X]^\omega$ is *M-stationary* if, for all club $E \in M$, $S \cap M \cap E \neq \emptyset$. A set mapping Σ is *open stationary* if $\Sigma(M)$ is open and M-stationary for all M.

Notice that open subsets of $[X]^\omega$ which are stationary are trivial in the sense that their complements are closed and not cofinal in $\langle [X]^\omega, \subseteq \rangle$. But it is not difficult to show that there are, for a given M, $\Sigma_0, \Sigma_1 \subseteq [M \cap X]^\omega$ which have empty intersection and which are each open and M-stationary.

Example 5.3. Let $X = \omega_1$. For each $M \prec H(\theta)$, choose $\alpha < M \cap \omega_1$ and set $\Sigma(M) = \{\gamma \in [\omega_1]^\omega : \alpha \in \gamma \subseteq M \cap \omega_1\}$. Then Σ is trivially open stationary.

Definition 5.4. We say a sequence of sets indexed by ordinals is a *continuous $\in$-chain* if it is $\subseteq$-continuous and $\in$-increasing.

Definition 5.5. An open stationary set mapping Σ *reflects* if there exists a continuous $\in$-chain $\langle N_\xi : \xi < \omega_1 \rangle$ such that, for all $\nu < \omega_1$, $N_\nu \in \mathrm{dom}(\Sigma)$ and there exists $\nu_0 < \nu$ such that $N_\xi \cap X \in \Sigma(N_\nu)$ for all ξ satisfying $\nu_0 < \xi < \nu$. We say that such an $\langle N_\xi : \xi < \omega_1 \rangle$ is a *reflecting sequence* for Σ.

Definition 5.6. The *Set Mapping Reflection Principle (*MRP*)* is the assertion that all open stationary set mappings reflect.

One can view MRP as asserting that every open stationary Σ contains a copy of Example 5.3.

Theorem 5.7. MRP *implies the existence of a well ordering of $\mathcal{P}(\omega_1)/NS$ which is definable over $(H(\aleph_2), \in)$ with parameters.*

Definition 5.8. Given $A, B \subseteq \omega_1$, define $A \equiv_{\mathrm{NS}} B$ to mean $A \triangle B$ is non stationary.

The following fact follows easily from the existence of a partition of ω_1 into ω_1 pairwise disjoint stationary sets.

Fact 5.9. *There exist $2^{\aleph_1}$-many $\equiv_{NS}$-equivalence classes.*

Proof. (of Theorem 5.7) Fix $\langle C_\delta : \delta \in \mathrm{Lim}(\omega_1) \rangle$ such that, for all δ, $\mathrm{otp}(C_\delta) = \omega$ and C_δ is cofinal in δ. If $A \subseteq B \in [\mathrm{O}rd]^{\leq \omega}$, $\sup A < \sup B$, and B has no maximum, then set $w(A, B) = |\pi^{-1}(C_\delta) \cap \sup A|$ where $\delta = \mathrm{otp}(B)$ and π is the unique order isomorphism from B to δ. Note that $w(A, B)$ is necessarily finite.

Set $X = \omega_2$ and $\theta = \left(2^{2^{\aleph_1}}\right)^+$. Given $M \prec H(\theta)$, define $\Sigma_<(M)$ to be the set of $A \in [\omega_2]^\omega$ for which the following conditions hold:

$$\sup(A \cap \omega_1) < \sup(M \cap \omega_1),$$

$$\sup A < \sup(M \cap \omega_2),$$
$$w(A \cap \omega_1, M \cap \omega_1) < w(A, M \cap \omega_2).$$

Analogously define $\Sigma_{\geq}(M)$ with $\geq$ replacing $<$ in the last inequality. Intuitively, $\Sigma_{<}(M)$ consists of those countable subsets of $M \cap \omega_2$ whose intersection with ω_2 is "higher" in $M \cap \omega_2$ than its intersection with ω_1 is in $M \cap \omega_1$.

Observe that $\Sigma_{<}(M)$ and $\Sigma_{\geq}(M)$ are open. To see this, suppose that $A \subseteq M \cap \omega_2$ with

$$\sup(A \cap \omega_1) < \sup(M \cap \omega_1),$$
$$\sup A < \sup(M \cap \omega_2),$$

and let α and β be the least elements of $A \cap \omega_1$ and A, respectively, such that

$$w(\alpha, M \cap \omega_1) = w(A \cap \omega_1, M \cap \omega_1)$$
$$w(A \cap \beta, M \cap \omega_2) = w(A, M \cap \omega_2)$$

If A is in $\Sigma_{<}(M)$, then $[\{\alpha, \beta\}, A] \subseteq \Sigma_{<}(M)$ and similarly for $\Sigma_{\geq}(M)$.

Claim 5.10. *$\Sigma_{<}(M)$ is M-stationary.*

Proof. Let $E \in M$ be club subset of $[\omega_2]^\omega$. Observe that there exists $\alpha < \omega_1$ such that $\{\sup A : A \in E \wedge A \cap \omega_1 = \alpha\}$ is cofinal in ω_2. Let $\alpha \in M$ be as above. Let $n = w(\alpha, M \cap \omega_1)$. Let $\beta \in M \cap \omega_2$ satisfy $w(\beta \cap M, \omega_2 \cap M) > n$. Pick $A \in E$ such that $A \cap \omega_1 = \alpha$ and $\sup A > \beta$. Since $E \in M$, we may assume $A \in M$. Thus, $A \in \Sigma_{<}(M) \cap M \cap E$. □

Claim 5.11. *$\Sigma_{\geq}(M)$ is M-stationary.*

Proof. Let $E \in M$ be club subset of $[\omega_2]^\omega$. Let $N \in M$ satisfy N be an elementary submodels of $H(2^{\aleph_1 +})$ with $|N| = \aleph_1$, and $\{E\} \cup \omega_1 \subseteq N$. Observe that by elementarity of M, $\sup(N \cap \omega_2)$ is an element of M and hence $\sup(N \cap M \cap \omega_2) < \sup(M \cap \omega_2)$. Set

$$n = w(N \cap M \cap \omega_2, M \cap \omega_2)$$

and $E_0 = E \cap N$. Then $w(A \cap \omega_2, M \cap \omega_2) \leq n$ for all $A \in E_0 \cap M$. Moreover, E_0 is club in $[\omega_2 \cap N]^\omega$; hence, $\{\sup(A \cap \omega_1) : A \in E_0 \cap M\}$ is unbounded in $M \cap \omega_1$. Hence, there exists $A \in E_0 \cap M$ such that $w(A \cap \omega_1, M \cap \omega_1) \geq n$. Hence, $\Sigma_{\geq}(M) \cap M \cap E \neq \emptyset$. □

For each $A \subseteq \omega_1$, let $\Sigma_A(M) = \Sigma_{<}(M)$ if $M \cap \omega_1 \in A$ and $\Sigma_A(M) = \Sigma_{\geq}(M)$ if $M \cap \omega_1 \notin A$. Let $\langle N_\xi : \xi < \omega_1 \rangle$ reflect Σ_A. Set $\delta = \bigcup_{\xi < \omega_1} N_\xi \cap \omega_2$. Since $\langle N_\xi : \xi < \omega_1 \rangle$ is a continuous $\in$-chain of elementary submodels of $H(\theta)$, we have $\omega_1 \subseteq \bigcup_{\xi < \omega_1} N_\xi \prec H(\theta)$ and $\{N_\xi \cap \omega_2 : \xi < \omega_1\}$ a club subset of $[\delta]^\omega$. Hence, δ is an ordinal such that $\mathrm{cf}(\delta) = \omega_1 < \delta < \omega_2$. Moreover, δ satisfies the following property $\phi(A, \delta)$:

> there is a club $\mathcal{M} \subseteq [\delta]^\omega$ which is well ordered by $\subseteq$ and is such that for all limit $\nu < \omega_1$ there is a $\nu_0 < \nu$ with ν is in A if and only if
>
> $$w(M_\xi \cap \omega_1, M_\nu \cap \omega_1) < w(M_\xi, M_\nu)$$
>
> whenever $\nu_0 < \xi < \nu$.

Here M_ξ is the ξ^{th} element of $\mathcal{M}$ in its $\subseteq$-increasing enumeration. If we let δ_A be the least ordinal such that $\phi(A, \delta)$ holds, then $A \mapsto \delta_A$ is definable over $H(\aleph_2)$ with parameter $\langle C_\nu : \nu \in \mathrm{Lim}(\omega_1) \rangle$. Hence it suffices to prove the following claim.

Claim 5.12. *If A and B are subsets of ω_1 and $\phi(A,\delta) \wedge \phi(B,\delta)$ holds for some δ, then $A \equiv_{NS} B$.*

Proof. First observe that if $\mathcal{N}$ is a club witnessing $\phi(A,\delta)$ and $\mathcal{N}' \subseteq \mathcal{N}$ is also club, then $\mathcal{N}'$ also witnesses $\phi(A,\delta)$. Hence if $\phi(A,\delta) \wedge \phi(B,\delta)$, there is a single club $\langle N_\xi : \xi < \omega_1 \rangle$ in $[\delta]^\omega$ which witnesses both $\phi(A,\delta)$ and $\phi(B,\delta)$. Let $E = \{N_\xi \cap \omega_1 : \xi < \omega_1\}$. It suffices to show that no limit point of E is in $A \triangle B$. To see this, let ν be a limit ordinal. $N_\nu \cap \omega_1$ is in A iff there are arbitrarily large $\xi < \nu$ such that

$$w(N_\xi \cap \omega_1, N_\nu \cap \omega_1) < w(N_\xi \cap \omega_2, N_\nu \cap \omega_2)$$

iff $N_\nu \cap \omega_1$ is in B. □

This also completes the proof of the theorem. □

Remark. The main result of [3] shows that the coding in the proof of Theorem 5.7 above necessarily yields $2^{\aleph_0} = 2^{\aleph_1}$. Notice, however, that the forcing in Theorem 6.7 used to reflect an open stationary set mapping does not introduce new reals.

There is an analogous proof that SRP implies $2^{\aleph_1} = \aleph_2$ which can be described as follows. Suppose S is a stationary co-stationary subset of ω_1. For each $A \subseteq \omega_1$, set

$$\Gamma_A = \{X \in [\omega_2]^\omega : X \cap \omega_1 \in A \leftrightarrow \mathrm{otp}(X) \in S\}.$$

Woodin's statement ψ_{AC} is the assertion that for every $A \subseteq \omega_1$ and for every stationary co-stationary $S \subseteq \omega_1$, there is a $\delta < \omega_2$ of cofinality ω_1 and a club E in $[\delta]^\omega$ which is contained in Γ_A. Notice that for a given δ of cofinality ω_1, if Γ_A and Γ_B both contains a club in $[\delta]^\omega$ for $A, B \subseteq \omega_1$, then A and B differ by a non stationary set. Hence ψ_{AC} implies $2^{\aleph_1} = \aleph_2$.

6. PFA implies MRP

Let Q be a forcing (*i.e.*, a poset with a maximum element). For us, $p \leq q$ means p is stronger than q. The smallest θ for which "$G \subseteq Q$ is generic over $H(\theta)$" makes sense is $\theta = |2^Q|^+$, assuming the underlying set of Q is $|Q|$.

Definition 6.1. Q is *proper* if, for all sets X, forcing with Q preserves all stationary subsets of $[X]^\omega$.

The following characterization due to Jech provides the standard method for verifying a forcing Q is proper. In order to state this characterization in a concise manner, it is helpful to make the following additional definition.

Definition 6.2. Suppose that Q is a forcing and $M \prec H(|2^Q|^+)$ with Q in M. A condition $\bar{q}$ in Q is (M,Q)-*generic* if whenever $D \subseteq Q$ is a dense open set in M and $r \leq \bar{q}$, there is an element of $D \cap M$ compatible with r. Equivalently, $\bar{q}$ forces that $\dot{G} \cap M$ is generic over M (here $\dot{G}$ is the Q-name for the generic filter).

Proposition 6.3. *A forcing Q is proper if and only if whenever $\mathcal{P}(Q) \in M \prec H(\theta)$ and $q \in Q \cap M$, there exists $\bar{q} \leq q$ which is* (M,Q)-generic.

Remark. The assumption that $\mathcal{P}(Q)$ is in M is natural since then the collection of all dense open subsets of Q is an element of M. The meaning of the statement "$\mathcal{P}(Q)$ in M" should be clear but is somewhat subtle: we want the powerset of Q's underlying set (which we also denote by Q) to be in M, as well as the order on Q. In the above proposition we can also fix θ to be minimal with the property that $\mathcal{P}(Q)$ is in $H(\theta)$.

Definition 6.4. The Proper Forcing Axiom (PFA) is the assertion that if Q is proper and D_ξ is predense in Q for all $\xi < \omega_1$, then there exists $G \subseteq Q$ such that G is a filter and $G \cap D_\xi \neq \emptyset$ for all $\xi < \omega_1$.

The following two classes of forcings and their iterations (which are also proper by a well known theorem of Shelah [17]) already are sufficient to yield many of the consequences of PFA (including the failure of $\square(\theta)$ and $2^{\aleph_0} = \aleph_2$). For more about proper forcing, see [6], [7], [17], [21].

Example 6.5. Every c.c.c. forcing is proper. To see this, note that the definition of (M, Q)-generic remains unchanged if one replaces "dense open" with "maximal antichain." It then follows from Fact 3.7, that every condition in a c.c.c. forcing Q is (M, Q)-generic for any relevant M.

Example 6.6. Every σ-closed forcing is proper. To see this, start with $q_0 = q \in M$. Let $\{A_n\}_{n<\omega}$ enumerate all predense sets in M. Construct $\langle q_n : n < \omega\rangle$ decreasing in $Q \cap M$ such that q_{n+1} is below an element of $A_n \cap M$. Any lower bound for $\{q_n\}_{n<\omega}$ is an (M, Q)-generic condition and such a lower bound exists by assumption that Q is σ-closed.

The proof of the following theorem is a standard verification of properness.

Theorem 6.7. PFA *implies* MRP*.*

Proof. Let Σ be an open stationary set mapping (with X and θ as before). Let Q be the set of all continuous $\in$-chains $\langle N_\xi : \xi \leq \alpha\rangle$ in $\mathrm{dom}(\Sigma)$ for which $\alpha < \omega_1$ and, for all limit $\nu \leq \alpha$, there exists $\nu_0 < \nu$ such that $N_\xi \cap X \in \Sigma(N_\nu)$ for all $\xi \in \nu \setminus \nu_0$. That is Q consists of all countable partial reflecting sequences for Σ. Order Q by extension. The following claim implies that it suffices to prove that Q is proper.

Claim 6.8. *Given that* Q *is proper,* $\{q \in Q : \alpha \in \mathrm{dom}(q)\}$ *is dense for all* $\alpha < \omega_1$*.*

Proof. The set $\{q \in Q : x \in \bigcup \mathrm{ran}(q)\}$ is dense for all $x \in H(\theta)$; hence, $\mathbb{1} \Vdash \check{H}(\theta) = \bigcup_{q \in \dot{G}} \bigcup \mathrm{ran}(q)$. Since Q is proper, it does not collapse ω_1, so, since $H(\theta)$ is uncountable, $\mathbb{1} \Vdash \alpha \in \mathrm{dom}(\bigcup \dot{G})$ for all $\alpha < \omega_1$. □

Set $\lambda = 2^{<\theta}$ and let $\Sigma, Q \in M \prec H(2^{\lambda^+})$. Fix $q_0 \in Q \cap M$. Observe that if $\langle q_n : n < \omega\rangle$ is a decreasing sequence in $Q \cap M$ such that each q_{n+1} is below some element of D_n, the nth dense subset of Q in M, and $q_\omega = \bigcup_{n<\omega} q_n = \langle N_\xi : \xi < \alpha\rangle$, then $\bigcup_{\xi<\alpha} N_\xi = M \cap H(\theta)$ because, for all $x \in M \cap H(\theta)$, the set of $p \in Q$ such that $x \in \bigcup \mathrm{ran}(p)$ is dense in Q. So, to prove Q is proper, it suffices to show that $N_\xi \cap X \in \Sigma(M \cap H(\theta))$ for all $\xi \in \alpha \setminus \mathrm{dom}(q_0)$, for then $q_\omega \cup \{\langle \alpha, M \cap H(\theta)\rangle\}$ will be an (M, Q)-generic element of Q below q_0. Therefore, it suffices to show that, given $q_n \in Q \cap M$, there exists $q_{n+1} \in D_n \cap M$ such that $q_{n+1} \leq q_n$ and $q_{n+1}(\xi) \cap X \in \Sigma(M \cap H(\theta))$ for all $\xi \in \mathrm{dom}(q_{n+1}) \setminus \mathrm{dom}(q_n)$.

Using M-stationarity of $\Sigma(M \cap H(\theta))$, let $N \in M$ satisfy $D_n, q_n, Q, \Sigma \in N \prec H(\lambda^+)$ and $N \cap X \in \Sigma(M \cap H(\theta))$. Using openness of $\Sigma(M \cap H(\theta))$, let $a \in [N \cap X]^{<\omega}$ satisfy $[a, N \cap X] \subseteq \Sigma(M \cap H(\theta))$. Set $q = q_n \cup \{\langle \mathrm{dom}(q_n), P\rangle\}$ where $P \in N \cap \mathrm{dom}(\Sigma)$ and $q_n(\max(\mathrm{dom}(q_n))) \cup a \subseteq P$. Since $q, D_n, Q \in N$, there exists $q_{n+1} \in D_n \cap N$ such that $q_{n+1} \leq q$. Since $D_n, q, Q, N \in M$, we may assume $q_{n+1} \in M$. Since $q_{n+1} \in N$ and $a \subseteq P$, every $\xi \in \mathrm{dom}(q_{n+1}) \setminus \mathrm{dom}(q_n)$ satisfies $q_{n+1}(\xi) \in [a, N \cap X]$. □

Now consider the following strengthening of properness.

Definition 6.9. A forcing Q is ω-proper if, given $q \in Q$ and an $\in$-chain $\langle N_i : i < \omega \rangle$ of elementary submodel of $H(|2^Q|^+)$ such that $q, Q \in N_0$, there exists $\bar{q} \leq q$ such that $\bar{q}$ is (N_i, Q)-generic for all $i < \omega$.

To see the relevance of ω-proper, consider the club of those $M \prec H(\theta^+)$ (for some fixed θ) which is a union of an increasing chain of $M_i \prec M$ of elements of M. If N is an elementary submodel of $H(2^{\theta^+})$ and we define

$$\Sigma(N) = [H(\theta^+)]^\omega \setminus \{N_i : i < \omega\}$$

where N_i is the increasing sequence chosen for $N \cap H(\theta^+)$, then any forcing which reflects Σ, can not be ω-proper. This is so even if we weaken ω-properness to *weak ω-properness* where we require only the existence of a $\bar{q}$ which forces that

$$\{i < \omega : \dot{G} \cap N_i \text{ is } N_i\text{-generic}\}$$

is infinite (this definition is due to Eisworth and Nyikos [4]). This observation can be cast into a theorem (due to Shelah) as follows.

Theorem 6.10. *(Weakly) ω-proper forcings preserve (weak) club guessing sequences on ω_1. Moreover weakly ω-proper forcings preserve $\mho$-sequences.*

Proof. We will only prove the theorem for club guessing sequences. Let $\langle C_\alpha : \alpha < \omega_1 \rangle$ be club guessing. Let Q be ω-proper, $\dot{D}$ be a Q-name for a club subset of ω_1 and q be in Q. It suffices to find an extension of q which forces that $\dot{D}$ contains some C_α. Let $\langle M_\xi : \xi < \omega_1 \rangle$ be a continuous $\in$-chain in $H(|2^Q|^+)$ such that $Q \in M_0$. Set

$$E = \{\xi < \omega_1 : \omega_1 \cap M_\xi = \xi\}.$$

Hence, $C_\delta \subseteq E$ for some $\delta < \omega_1$. Let $N_i = M_{\xi_i}$ where $\langle \xi_i : i < \omega \rangle$ is an increasing enumeration of C_δ. Then every $\bar{q}$ that is (N_i, Q)-generic for $i < \omega$ forces $C_\delta \subseteq \dot{D}$ as desired. □

It is worth noting that the ϵ-collapse forcing of Baumgartner (which forms the cornerstone of Todorcevic's "models as side conditions" method [19], [21]), can also be shown to be weakly ω-proper. Very few applications of PFA prior to [14] required more than the forcing axiom for weakly ω-proper forcings.

7. Influence of MRP on the club filter

In this section we will consider how the assumption MRP influences the combinatorics of the club filter. The first is hardly more than an observation.

Definition 7.1. If X and Y are countable subsets of ω_1 which are closed in their supremum, then we say X *measures* Y if there is a $\xi < \sup X$ such that $X \setminus \xi$ is either contained in or disjoint from Y.

We define *measuring* to be the assertion that for every sequence $\langle D_\alpha : \alpha < \omega_1 \rangle$ with $D_\alpha \subseteq \alpha$ closed for all $\alpha < \omega_1$, there is a club $E \subseteq \omega_1$ such that $E \cap \alpha$ measures D_α whenever α is a limit point of E. Notice that measuring implies the non existence of $\mho$-sequences: if $\langle f_\alpha : \alpha < \omega_1 \rangle$ is a $\mho$-sequence, then for any i, $\langle f_\alpha^{-1}(i) : \alpha < \omega_1 \rangle$ is not measured by any club.

Theorem 7.2. MRP *implies measuring.*

Proof. This is not hard to verify; we will prove a more general statement in the next section. □

We will justify the name "measuring" momentarily. First it will be helpful make a definition and prove a few claims.

Definition 7.3. If $M \prec H(\theta)$, we say that a club $E \subseteq \omega_1$ *diagonalizes M's club filter* if $\delta = M \cap \omega_1$ is a limit point of E and whenever $D \subseteq \omega_1$ is a club in M, there is a $\delta_0 < \delta$ such that $E \cap (\delta_0, \delta) \subseteq D$.

Claim 7.4. *If $\langle N_\xi : \xi < \omega_1 \rangle$ is a continuous $\in$-chain of countable elementary submodels of some $H(\theta)$ for $\theta \geq \omega_2$, then $E = \{N_\xi \cap \omega_1 : \xi < \omega_1\}$ diagonalizes the club filter of N_ν whenever $\nu < \omega_1$ is a limit ordinal.*

Proof. If ν is a limit ordinal and $D \subseteq \omega_1$ is a club in N_ν, then by continuity of the sequence there is a $\xi < \nu$ such that D is in N_ξ. By Fact 3.8, $N_\eta \cap \omega_1$ is in D whenever $\xi < \eta$. It follows that if $\delta = N_\nu \cap \omega_1$ and $\delta_0 = N_\xi \cap \omega_1$, then

$$E \cap (\delta_0, \delta) = \{N_\eta \cap \omega_1 : \xi < \eta < \nu\}$$

is contained in D. □

Claim 7.5. *If E diagonalizes the club filter of M, then $[\omega_1]^\omega \setminus E$ is M-stationary.*

Proof. Suppose $D \subseteq \omega_1$ is a club in M. Since the limit points D' of D is also a club in M, $(D \setminus D') \cap E$ is bounded in $M \cap \omega_1$ and hence $D \setminus (D' \cap E)$ is non-empty. □

Proposition 7.6. *The following are equivalent:*

(1) *Measuring holds.*
(2) *If M is a countable elementary submodel of $H(\aleph_2)$ and $Y \subseteq M \cap \omega_1$ is closed and in* Hull$(M \cup \{M \cap \omega_1\})$*, then there is a club $E \subseteq \omega_1$ in M such that $E \cap M$ is either contained in or disjoint from Y (i.e. Y is measured by the club filter of M).*

Proof. To see the forward implication, suppose measuring holds and let M and Y be given. Since Y is in the Skolem hull of $M \cup \{M \cap \omega_1\}$, there is a function f in M defined on ω_1 such that $f(M \cap \omega_1) = Y$. Without loss of generality, $f(\alpha)$ is a closed subset of α for each $\alpha < \omega_1$. Applying measuring in M, there is a club $E \subseteq \omega_1$ such that $E \cap \alpha$ measures $f(\alpha)$ for each α which is a limit point of E. By elementarity, $M \cap \omega_1$ is a limit point of E. By removing an initial part of E if necessary, we may assume that $E \cap M$ is either contained in or disjoint from Y.

To see the reverse implication, suppose $\langle Y_\alpha : \alpha < \omega_1 \rangle$ is a sequence such that for all $\alpha < \omega_1$, Y_α is a closed subset of α.

Let $\langle N_\xi : \xi < \omega_1 \rangle$ be a continuous $\in$-chain of countable elementary submodels such that $\langle Y_\alpha : \alpha < \omega_1 \rangle$ is in N_0. Let $E = \{N_\xi \cap \omega_1 : \xi < \omega_1\}$. By Claim 7.4, E diagonalizes the club filter of N_ν whenever ν is a limit ordinal. Also, if δ is a limit point of E, then there is a limit ordinal ν such that $N_\nu \cap \omega_1 = \delta$. By our assumption (2), there is a club D in N_ν which is either contained in or disjoint from Y_δ. It follows that $E \cap \delta$ measures Y_δ. □

Measuring is arguably the simplest consequence of PFA which is not known to be (in)consistent with CH.

Problem 7.7. Is measuring consistent with CH?

Eisworth and Nyikos have shown that measuring for sequences of clopen $Y_\alpha \subseteq \alpha$ is consistent with CH [4]. It is similarly known by [17] that measuring for sequences $\langle Y_\alpha : \alpha < \omega_1 \rangle$ on which the ordertype function is regressive is consistent with CH.

Now we consider a coherence property of the club filter considered by Larson [11].

Definition 7.8. Let $(+)$ denote the statement that there exists a stationary $S \subseteq [H(\aleph_2)]^\omega$ such that, for all $M, M' \in S$, if $M \cap \omega_1 = M' \cap \omega_1$, then, for every $E \in M$ and $E' \in M'$ such that E and E' are club subsets of ω_1, the set $E \cap E' \cap M \cap \omega_1$ is cofinal in $M \cap \omega_1$. Let $(-)$ denote $\neg(+)$.

Larson[11] showed that $(+)$ follows from club guessing on ω_1. Very recently Tetsuya Ishiu has shown that $(+)$ is consistent with the failure of club guessing (even in the presence of CH).

Theorem 7.9. MRP *implies* $(-)$.

Proof. Fix a stationary set S. We will show that S does not witness $(+)$. Suppose that M is such that $S \in M \prec H({2^{\aleph_1}}^+)$. Ask:

$(*)$ Is there an end extension $\widetilde{M}$ of M such that $\widetilde{M} \cap H(\aleph_2) \in S$ and there exists a club $E_M \subseteq \omega_1$ such that $E_M \in \widetilde{M}$ and E_M diagonalizes the club filter of M?

If "no," then set $\Sigma(M) = [\omega_1]^\omega$. If "yes," then set $\Sigma(M) = [\omega_1]^\omega \setminus E_M$ for some such E_M. In either case, $\Sigma(M)$ is open. Moreover, $\Sigma(M)$ is M-stationary by the above observation.

Suppose $\langle N_\xi : \xi < \omega_1 \rangle$ reflects Σ. Let $\langle N_\xi : \xi < \omega_1 \rangle \in \overline{M} \prec H({2^{\aleph_1}}^+)$ and $\overline{M} \cap H(\aleph_2) \in S$. By Fact 4.1, $\overline{M}$ end extends N_δ where $\delta = \overline{M} \cap \omega_1$. Also $E = \{N_\xi \cap \omega_1 : \xi < \omega_1\}$ is in $\overline{M}$ and by Claim 7.4 diagonalizes the club filter of N_δ. Hence the answer to $(*)$ is "yes" for N_δ. Therefore there exist an end extension $\tilde{M} \in S$ of N_δ and a club $E' \in \tilde{M}$ such that E' diagonalizes the club filter of N_δ and $\Sigma(N_\delta) = [\omega_1]^\omega \setminus E'$. Hence, for some $\xi < \delta$, we have $N_\nu \cap \omega_1 \notin E'$ for all $\nu \in \delta \setminus \xi$; hence, $E \cap E' \cap \delta$ is bounded in δ. The sets $\overline{M} \cap H(\aleph_2)$ and $\tilde{M} \cap H(\aleph_2)$ now demonstrate that S does not witness $(+)$. □

8. The influence of MRP beyond $H(\aleph_2)$

In this section we will study the influence of MRP on sets higher up in the cumulative hierarchy. It is based on work of Viale.

Let κ be an uncountable regular cardinal. Suppose $\mathcal{I}$ is an ideal of closed subsets of κ (in the order topology). *I.e.*, for all $I_0, I_1 \in \mathcal{I}$, we have I_0 closed, every closed subset of I_0 in $\mathcal{I}$, and $I_0 \cup I_1 \in \mathcal{I}$. Consider the following three conditions on $\mathcal{I}$:

(I1) If $\beta < \kappa$, then there exists $\mathcal{J} \in [\mathcal{I}]^{\leq\omega}$ such that $\beta + 1 = \bigcup \mathcal{J}$.

(I2) If $X \in [\kappa]^\omega$, then $\mathcal{I} \restriction X$ is countably generated. *I.e.*, there exists $\mathcal{J} \in [\mathcal{I}]^\omega$ such that, for all $I \in \mathcal{I}$, there exists $J \in \mathcal{J}$ such that $I \cap X \subseteq J \cap X$.

(I3) If $Z \subseteq \kappa$ is unbounded, then there exists $Y \in [Z]^\omega$ such that $Y \not\subseteq I$ for all $I \in \mathcal{I}$.

Recall that the Singular Cardinals Hypothesis (SCH) is the assertion that if λ is a singular strong limit cardinal, then $2^\lambda = \lambda^+$. By a theorem of Silver [18], if SCH fails at λ and $\mathrm{cf}(\lambda)$ is uncountable, then there is a stationary set of singular $\mu < \lambda$ such that SCH fails at μ. In particular, if λ is the least singular cardinal such that $2^\lambda > \lambda^+$, then $\mathrm{cf}(\lambda) = \omega$.

Theorem 8.1. *If* SCH *fails, then there exist* κ *and* $\mathcal{I}$ *satisfying* (*I1*), (*I2*), *and* (*I3*).

Proof. Assume SCH fails. Then there exists μ such that $\omega = \mathrm{cf}(\mu) < \mu = 2^{<\mu}$ and $\mu^+ < 2^\mu$. Set $\kappa = \mu^+$ and let $\mu = \sup_{n<\omega} \mu_n$. For all $\beta < \kappa$, let $\beta + 1 = \bigcup_{n<\omega} E^\beta_n$ where $\langle E^\beta_n : n < \omega\rangle$ is an ascending sequence of closed subsets of κ such that $|E^\beta_n| \leq \mu_n$ for all n. By modifying the sequence, arrange that $E^\beta_{n+1} \supseteq \bigcup_{\alpha\in E^\beta_n \cap\beta} E^\alpha_n$ for all n. Let $\mathcal{I}$ be the ideal generated by $\{E^\beta_n : n < \omega \wedge \beta < \kappa\}$. Then (I1) is trivially true. For (I2), fix $X \in [\kappa]^\omega$. Then, for all $\beta < \kappa$, $\{E^\beta_n \cap X : n < \omega\}$ generates an ideal $\mathcal{I}^\beta_X$ on $\mathcal{P}(X)$. Moreover, if $\beta < \beta'$, then $\mathcal{I}^\beta_X \subseteq \mathcal{I}^{\beta'}_X$. Since $\mathrm{cf}(\kappa) > 2^{2^{\aleph_0}}$, as β increases, this restricted ideal eventually stabilizes to $\mathcal{I} \restriction X$. Therefore, $\{E^\beta_n \cap X\}_{n<\omega}$ generates $\mathcal{I} \restriction X$ if β is large enough. Finally, (I3) holds simply because $|\mathcal{I}| = 2^{<\mu} \cdot \mu^+ < 2^\mu = \mu^\omega = \mathrm{cf}(\langle[\mu^+]^\omega, \subseteq\rangle)$. □

Theorem 8.2. MRP *implies that there does not exist a regular cardinal κ and an ideal $\mathcal{I}$ on κ which satisfies (I1), (I2), and (I3).*

Proof. Seeking a contradiction, suppose κ and $\mathcal{I}$ satisfy (I1), (I2), and (I3). For each $\alpha \in \mathrm{Lim}(\omega_1)$, fix a cofinal $C_\alpha \subseteq \alpha$ such that $\mathrm{otp}(C_\alpha) = \omega$. Fix, for each $M \prec H(2^{\kappa^+})$, a $\subseteq$-increasing sequence $\langle I^M_n : n < \omega\rangle$ of elements of $\mathcal{I}$ which generates $\mathcal{I} \restriction M$. Let $\Sigma(M)$ be the set of $N \in [\kappa]^\omega$ for which $\sup(N \cap \omega_1) < M \cap \omega_1$ and $\sup N \notin I^M_n$ where $n = |C_{M\cap\omega_1} \cap N|$.

Claim 8.3. *$\Sigma(M)$ is open and M-stationary.*

Proof. To see that $\Sigma(M)$ is open, suppose that N is in $\Sigma(M)$ and let α be an element of $N \cap \omega_1$ which is an upper bound for $N \cap C_\delta$ where $\delta = M \cap \omega_1$. Since $\beta = \sup N$ is not in I^M_n, where $n = |C_\delta \cap N|$, there is a β_0 in N such that $(\beta_0, \beta) \cap I^M_n = \emptyset$. Then $[\{\alpha, \beta_0\}, N] \subseteq \Sigma(M)$.

To see that $\Sigma(M)$ is M-stationary, suppose $E \in M$ is a club subset of $[\kappa]^\omega$. Fix $\alpha < M \cap \omega_1$ such that $\Gamma = \{\sup A : A \in E \wedge A \cap \omega_1 = \alpha\}$ is cofinal in κ. Set $E' = \{A \in E : A \cap \omega_1 = \alpha\}$. Using (I3), pick $X \in [\Gamma]^\omega \cap M$ such that X is not contained in any $I \in \mathcal{I}$. Pick $\beta \in X \setminus I^M_n$ where $n = |C_\delta \cap \alpha|$ where $\delta = M \cap \omega_1$. Then $\beta \in M$ because $X \in M$ and X is countable. Pick $A \in E' \cap M$ such that $\sup A = \beta$. Then $A \in \Sigma(M) \cap M \cap E$. Thus, $\Sigma(M)$ is M-stationary. □

Suppose $\langle N_\xi : \xi < \omega_1\rangle$ is a reflecting sequence and let $E = \{\sup(N_\xi \cap \kappa) : \xi < \omega_1\}$. Using (I1), pick $I \in \mathcal{I}$ such that $I \cap E$ is cofinal in E. Let δ be such that $\Xi = \{\xi < \delta : \sup(N_\xi \cap \kappa) \in I\}$ is cofinal in δ. Then there exists n such that $I \cap N_\delta \subseteq I^{N_\delta}_n$. Pick $\xi \in \Xi$ such that $|C_{N_\delta\cap\omega_1} \cap N_\xi| \geq n$ and $N_\xi \cap \kappa \in \Sigma(N_\delta)$. Then $\sup(N_\xi \cap \kappa) \notin I^{N_\delta}_{|N_\xi\cap C_\delta|}$ and therefore not in $I^{N_\delta}_n$ by our assumption that $\langle I^{N_\delta}_k : k < \omega\rangle$ is $\subseteq$-increasing. Hence $\sup(N_\xi \cap \kappa) \notin I$, in contradiction with how we chose ξ. □

Corollary 8.4. MRP *implies* SCH.

Now recall the definition of $\square(\kappa)$.[2]

Definition 8.5. $\square(\kappa)$ asserts that there exists a sequence $\langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa)\rangle$ satisfying the following conditions:

(1) C_α is closed and cofinal in α.
(2) If $\alpha \in \mathrm{Lim}(C_\beta)$, then $C_\alpha = C_\beta \cap \alpha$.

[2] $\square_\kappa$ and $\square(\kappa)$ are different principles. They are related in the sense that $\square(\kappa^+)$ is a formal weakening of $\square_\kappa$.

(3) There is no closed and cofinal $E \subseteq \kappa$ such that $C_\beta = E \cap \beta$ for all $\beta \in \mathrm{Lim}(E)$.

We can similarly use this route to show that MRP implies $\square(\kappa)$ fails.

Theorem 8.6. [22] *If $\square(\kappa)$ holds, then there exists $\mathcal{I}$ satisfying (I1), (I2), and (I3).*

Proof. Fix a $\square(\kappa)$ sequence $\langle C_\alpha : \alpha < \kappa \rangle$ and, following [20], define

$$\varrho_2(\alpha, \alpha) = 0$$

$$\varrho_2(\alpha, \beta) = 1 + \varrho_2(\alpha, \min C_\beta \setminus \alpha)$$

whenever $\alpha < \beta < \kappa$. Notice that $\varrho_2(\alpha, \beta) \leq 1$ if and only if α is in C_β. It is sufficient to show that the ideal $\mathcal{I}$ generated by the sets

$$I_{\beta,n} = \{\alpha \in \beta : \varrho_2(\alpha, \beta) \leq n\}$$

satisfies (I1)-(I3). The following identity clarifies the relationship between $\mathcal{I}$ and the ideal generated by $\langle C_\alpha : \alpha < \kappa \rangle$:

$$I_{\beta,n+1} = I_{\beta,n} \cup \bigcup_{\gamma \in I_{\beta,n}} C_\gamma \setminus \sup(I_{\beta,n} \cap \gamma).$$

That $\mathcal{I}$ satisfies (I1) is trivial, since for every $\alpha < \beta$, $\varrho_2(\alpha, \beta) < \omega$. The following are standard properties of ϱ_2 [20]:

(1) for all $\alpha < \beta < \kappa$, the set $\{|\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)| : \xi < \alpha\}$ is bounded in ω.
(2) $\varrho_2[Z]^2$ is unbounded in ω for every Z unbounded in κ;

To see that $\mathcal{I}$ satisfies (I2), notice that, for all $\alpha < \beta < \kappa$ and $n < \omega$, we have $I_{\beta,n} \cap \alpha \subseteq I_{\alpha,k}$ for some $k < \omega$, and $I_{\alpha,k} \subseteq I_{\beta,l}$ for some $l < \omega$. Hence, if $X \in [\kappa]^\omega$ and $\beta \geq \sup(X)$, then $\{I_{\beta,n} \cap X : n < \omega\}$ generates $\mathcal{I} \restriction X$. For (I3), suppose $Z \subseteq \kappa$ is unbounded. Then $\varrho_2[Z]^2$ is unbounded in ω and in particular there is a $\beta < \kappa$ such that $\varrho_2[Z \cap \beta]^2$ is unbounded in ω. It follows that $Z \cap \beta$ is not contained in an element of $\mathcal{I}$. $\square$

For the sake of demonstration, we will now give a direct proof that MRP implies the failure of $\square$.

Theorem 8.7. *For all regular κ,* MRP *implies $\neg\square(\kappa)$.*

Proof. Suppose not; let $\langle C_\alpha : \alpha < \kappa \rangle$ witness $\square(\kappa)$. For all $M \prec H(2^{\kappa^+})$ such that $\langle C_\alpha : \alpha < \kappa \rangle \in M$, set $\Sigma(M)$ equal to the set of $N \in [\kappa]^\omega$ for which $\sup N < \sup(M \cap \kappa)$ and $\sup N \notin C_{\sup(M \cap \kappa)}$. Suppose $\langle N_\xi : \xi < \omega_1 \rangle$ reflects Σ. Set $E_0 = \{\sup(N_\xi \cap \kappa)\}_{\xi < \omega_1}$. Then E_0 is closed and cofinal in some δ with $\mathrm{cf}(\delta) = \omega_1$. Let $E = C_\delta \cap E_0$. Suppose $\nu \in \mathrm{Lim}(E)$. Let $\eta < \omega_1$ be such that $\nu = \sup(N_\eta \cap \kappa)$. Then $C_\nu = C_\delta \cap \nu$ and $\eta \in \mathrm{Lim}(\omega_1)$. Let $\eta_0 < \eta$ be such that $N_\xi \cap \kappa \in \Sigma(N_\eta)$ for all $\xi \in \eta \setminus \eta_0$. Let $\nu' \in E$ be such that $\sup(N_{\eta_0} \cap \kappa) \leq \nu' < \sup(N_\nu \cap \kappa)$. Then there exists $\xi \in \eta \setminus \eta_0$ such that $\nu' = \sup(N_\xi \cap \kappa)$. Since $N_\xi \cap \kappa \in \Sigma(N_\nu)$, we have $\sup(N_\xi \cap \kappa) \notin C_\nu$. But $\sup(N_\xi \cap \kappa) \in E \cap \nu \subseteq C_\delta \cap \nu = C_\nu$, which is absurd. Thus, it suffices to show that Σ is open stationary.

Fix $M \in \mathrm{dom}(\Sigma)$ and $N \in \Sigma(M)$. Then $\sup N \notin C_{\sup(M \cap \kappa)}$; hence, there exists $\alpha \in N$ such that $(\alpha, \sup N) \cap C_{\sup(M \cap \kappa)} = \emptyset$; hence, $[\{\alpha\}, N] \subseteq \Sigma(M)$. Thus, $\Sigma(M)$ is open. Next, let $E \in M$ be a club subset of $[\kappa]^\omega$. Set $\Gamma = \{\sup A : A \in E\}$, which is closed and cofinal in κ. Then it suffices to show that $\Gamma \cap M \not\subseteq C_{\sup(M \cap \kappa)}$. Seeking a contradiction, suppose $\Gamma \cap M \subseteq C_{\sup(M \cap \kappa)}$. Then, for all $\{\alpha < \beta\} \in [\mathrm{Lim}(\Gamma \cap M)]^2$, we have $C_\alpha = C_{\sup(M\cap\kappa)} \cap \alpha$ and $C_\beta = C_{\sup(M\cap\kappa)} \cap \beta$; whence, $C_\alpha = C_\beta \cap \alpha$. By elementarity, we have $C_\alpha = C_\beta \cap \alpha$ for all $\{\alpha < \beta\} \in [\mathrm{Lim}(\Gamma)]^2$. Therefore, $C = \bigcup_{\alpha\in\mathrm{Lim}(\Gamma)} C_\alpha$ is closed and cofinal in κ and $C_\alpha = C \cap \alpha$ for all $\alpha \in \mathrm{Lim}(C)$, in contradiction with (3) of the definition of $\square(\kappa)$. $\square$

9. The 0-1 law for open set mappings

Let Σ be an open set mapping (for some X, θ). Are there some conditions that, in the presence of MRP or some stronger assumption, ensure that every $\Sigma(M)$ is trivial, *i.e.*, either $\Sigma(M)$ is not M-stationary or contains $E \cap M$ for some club $E \subseteq [X]^\omega$? Note that $\Sigma_<$ and $\Sigma_\geq$ are an example of nontriviality since they are everywhere disjoint open stationary set mappings.

Definition 9.1. The *0-1 law* for open set mappings asserts that, if Σ is an open set mapping defined on a club in $H(\theta)$ and for all $M \in \mathrm{dom}(\Sigma)$,

(1) for all $P \in \Sigma(M)$ and all end extensions $\overline{P}$ of P, we have $\overline{P} \in \Sigma(M)$, and,
(2) for all end extensions $\overline{M}$ of M in $\mathrm{dom}(\Sigma)$, we have $\Sigma(\overline{M})\cap M = \Sigma(M)\cap M$,

then there exists a club $E^* \subseteq \mathrm{dom}(\Sigma)$ such that, for all $M \in E^*$, there exists a club $E \subseteq [X]^\omega$ such that $E \in M$ and either $E \cap M \subseteq \Sigma(M)$ or $E \cap M \cap \Sigma(M) = \emptyset$.

Example 9.2. [8] Let T be an ω_1-tree and B be the set of uncountable maximal chains in T. Suppose $\{t_{\delta,n}\}_{n<\omega}$ is the δ^{th} level of T. Define $\Sigma_n(M)$ to be the set of $N \in [H(\aleph_2)]^\omega$ for which either N is not an elementary submodel of $H(\aleph_2)$, or there exists $b \in N \cap B$ such that $t_{\delta,n} \restriction (N \cap \omega_1) \in b$ where $\delta = M \cap \omega_1$. If each Σ_n satisfies the 0-1 law, then T has at most $\aleph_1$-many branches. In fact more generally, if $\mathcal{B}$ is a collection of uncountable downward $<_T$-closed subsets of T which have pairwise countable intersection, then if each Σ_n satisfies the 0-1 law, $\mathcal{B}$ has at most $\aleph_1$ many elements. This latter statement is known as *Aronszajn tree saturation.* It is equivalent to the corresponding statement in which T is Aronszajn.

In order to prove the *0-1 law*, we will need a strengthening of MRP.

Definition 9.3. Let SMRP (Strong Mapping Reflection Principle) assert that if Σ is an open stationary set mapping on some domain $S \subseteq [H(\theta)]^\omega$ that is not necessarily club, then there is a strong reflecting sequence $\langle N_\xi : \xi < \omega_1\rangle$ for S such that, for all $\nu \in \mathrm{Lim}(\omega_1)$, if $N_\nu \in S$, then there exists $\nu_0 < \nu$ such that $N_\xi \cap X \in \Sigma(N_\nu)$ for all $\xi \in \nu \setminus \nu_0$.

MM implies SMRP. If $\mathrm{dom}(\Sigma)$ is a club, then SMRP is just MRP. If Σ is always trivial, then SMRP is just SRP.

Theorem 9.4. SMRP *implies the 0-1 law for open set mappings.*

Proof. Let Σ satisfying (1) and (2) be given. Let S denote the set of $M \in \mathrm{dom}(\Sigma)$ for which $\Sigma(M)$ is M-stationary. Let $\langle N_\xi : \xi < \omega_1\rangle$ strongly reflect S and reflect Σ in the sense of SMRP. Set

$$E^* = \{M \in \mathrm{dom}(\Sigma) : \langle N_\xi : \xi < \omega_1\rangle \in M\}.$$

Let $M \in E^*$ be arbitrary. If $\Sigma(M)$ is not M-stationary, then there is trivially a club $E \subseteq [X]^\omega$ such that $E \in M$ and $E \cap M \cap \Sigma(M) = \emptyset$. Therefore, we may assume $\Sigma(M)$ is M-stationary. Then $N_\delta \in S$ where $\delta = M\cap\omega_1$ because $M\cap\omega_1 = N_\delta\cap\omega_1$. Let $\delta_0 < \delta$ be such that $N_\xi\cap X \in \Sigma(N_\delta)$ for all $\xi \in \delta\setminus\delta_0$. Define E_0 to be the set of all N in $[H(\theta)]^\omega$ such that $N\cap\omega_1$ is a limit ordinal and for all $\xi < N\cap\omega_1$, $N_\xi \subseteq N$

and $\delta_0 \in N$. If $N \in E_0 \cap M$, then N end extends N_ξ for some $\xi \in \delta \setminus \delta_0$. Since $\delta_0 < \xi$, $N_\xi \cap X \in \Sigma(N_\delta)$ and therefore $N_\xi \cap X \in \Sigma(M)$ by (2). Hence $N \cap X \in \Sigma(M)$ by (1). If we set $E = \{N \cap X : N \in E_0\}$, then E is a club subset of $[X]^\omega$, $E \in M$, and $E \cap M \subseteq \Sigma(M)$. $\square$

Remark. It is often the case that an open set mapping Σ satisfies the following additional hypothesis: if $\overline{N}$ is an end extension of N and $\Sigma(\overline{N})$ is $\overline{N}$-stationary, then $\Sigma(N)$ is N-stationary. This is satisfied, for instance, in the open set mappings defined in Example 9.2 and in [15]. It follows from the above argument that MRP implies the 0-1 law for open set mappings which satisfy this additional condition.

10. Open Problems

Here I have collected some open problems which are related to MRP.

While MRP implies $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$, the forcing to reflect a single open stationary set mapping does not introduce new reals. Many of these forcings are moreover *completely proper* (see Eisworth and Moore's lecture notes in this volume). It is not known whether the forcing axiom for completely proper forcings is consistent with CH and the corresponding fragments of MRP already seem to capture much of the generality of this question. An important special case of this is the following.

Problem 10.1. Is *measuring* consistent with CH?

Similarly, we would like to know how much of MRP is consistent with $2^{\aleph_0} > \aleph_2$. Aspero and Mota have announced that *measuring* is consistent with $2^{\aleph_0} > \aleph_2$.

A common theme with many of SRP's consequences is that they are typical examples of what MM implies but PFA was not known to imply. Most of the remaining consequences of SRP can not follow from PFA because, in general, they can not be forced with a proper forcing notion (in contrast to PFA). Some, however, follow from MRP via a proof similar to that used in the case of SRP. The following is an open problem in this vein.

Problem 10.2. Does MRP imply that there is a Q and a generic elementary embedding $j : V \to M \subseteq V^Q$ such that $\mathrm{crit}(j) = \omega_1$ and $M^{<\omega_1} \subseteq M$?

If NS_{ω_1} is saturated, then there is such a generic embedding if we take $Q = \mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$. In particular, the problem has a positive answer if one replaces MRP with SRP. Also, if there is a Woodin cardinal, then the "countable tower" $\mathbb{Q}_{<\delta}$ is such a forcing (see [10]). This latter example makes it very difficult to prove a negative answer to this question. While this would not serve to give a better lower bound on the consistency strength of PFA, it would be interesting if there are other ways to establish lower bounds on the consistency strength of PFA which do not involve the failure of $\square$ (and are at least at the level of a Mahlo cardinal).

Problem 10.3. What is the consistency strength of the *0-1 law for open set mappings*?

The *0-1 law* implies that there are no Kurepa trees and hence that ω_2 is inaccessible to subsets of ω_1. This seems to be the best known lower bound.

Let Q_0 be a forcing of size $\aleph_1$ and $\mathcal{A}$ be a collection of $\aleph_1$ many maximal antichains of Q_0. We say that Q_0 $\mathcal{A}$*-embeds* into Q if there is a injection $f : Q_0 \to Q$ which preserves order, compatibility, incompatibility, and the maximality of elements of $\mathcal{A}$. Many problems concerning applications of forcing axioms reduce to the question of when a given pair $(Q_0, \mathcal{A})$ as above can be $\mathcal{A}$-embedded into a proper forcing Q. In [15], MRP was useful in providing an answer to this in a special case. It seems natural (though ambitious) to ask if it is possible to use MRP to prove a general result of this form.

Problem 10.4. Assume MRP and let Q_0 be a partial order and $\mathcal{A}$ a collection of maximal antichains of Q_0 such that $|Q_0|, |\mathcal{A}| \leq \aleph_1$. Is there an informative necessary and sufficient condition for when Q_0 can be $\mathcal{A}$-embedded into a proper forcing Q?

For instance, is there an upper bound on the cardinality of such Q which is expressible in terms of the $\beth$ function? For those Q_0 which can be $\mathcal{A}$-embedded into a proper Q, is there a canonical form that Q can be assumed to take? Does the answer to these questions change if "proper" is replaced by "preserves NS_{ω_1}?"

11. Further reading

The notion of set mapping reflection was introduced in [14] in order to prove that BPFA implies that there is a well ordering of $\mathbb{R}$ which is definable over $H(\aleph_2)$ (and consequently that $L(\mathcal{P}(\omega_1))$ satisfies the Axiom of Choice). This paper also establishes that PFA implies MRP and that MRP implies the failure of $\square(\kappa)$ at all regular $\kappa > \omega_1$. Caicedo and Veličković [2] built on these ideas to show that BPFA implies there is a well ordering of $\mathbb{R}$ which is Δ_1-definable with parameters in $H(\aleph_2)$ (the complexity of the well ordering presented in Theorem 5.7 is Δ_2). Given the parameters, this complexity is optimal.

In [15], Moore used MRP in conjunction with BPFA to prove that every Aronszajn line contains a Countryman suborder. The 0-1 law was isolated from that proof and serves as the sole use of MRP in that paper. In [8], König, Larson, Moore, and Veličković further analyzed the role of the 0-1 law [15]. While the end goal was to reduce the consistency strength of the results of [15], much of [8] concerns a study of Aronszajn tree saturation. In particular, it is shown that MRP implies A-tree saturation and that the conjunction of A-tree saturation and BPFA implies that every Aronszajn line contains a Countryman suborder. It is not known whether the existence of a non-saturated Aronszajn tree implies that there is an Aronszajn line with no Countryman suborder.

The analysis of Aronszajn tree saturation was used explicitly in [13] to establish the consistent existence of a universal Aronszajn line. Similar combinatorial arguments were used by Ishiu and Moore [5] to characterize—assuming PFA^+—when a linear order contains an Aronszajn suborder. In [12], Moore showed that some non-trivial application of MRP is needed for the results of [15]. (In particular, if every Aronszajn line contains a Countryman suborder, then club guessing fails). Viale [23] proved that MRP implies the Singular Cardinals Hypothesis.

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