An introduction to $\mathbb{P}_{\max}$ forcing

Paul B. Larson*

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0.1 Pre-introduction

These notes are an account of a six-hour lecture series I presented at Carnegie Mellon University on September 9, 2006, at the inaugural meeting of the Appalachian Set Theory series. My remarks (occasionally improvised) were faithfully transcribed by Peter LeFanu Lumsdaine and Yimu Yin, who then presented me with a rough draft of this article. Their account aimed to capture the feel of the discussions (including some direct quotations), and I've tried to preserve that as much as possible.

1 Introduction

The forcing construction $\mathbb{P}_{\max}$ was invented by W. Hugh Woodin in the early 1990's in the wake of his result that the saturation of the nonstationary ideal on ω_1 (NS_{ω_1}) plus the existence of a measurable cardinal implies that there is a definable counterexample to the Continuum Hypothesis (in particular, it implies that $\boldsymbol{\delta}^1_2 = \omega_2$, which implies $\neg$CH). These notes outline a proof of the Π_2 maximality of the $\mathbb{P}_{\max}$ extension, which we can state as follows.

Theorem 1 ([9]). *Suppose that there exist proper class many Woodin cardinals, $A \subseteq \mathbb{R}$, $A \in L(\mathbb{R})$, φ is Π_2 in the extended language containing two additional unary predicates, and in some set forcing extension*

$$\langle H(\omega_2), \in, \mathrm{NS}_{\omega_1}, A^* \rangle \models \varphi$$

(where A^* is the reinterpretation of A in this extension). Then

$$L(\mathbb{R})^{\mathbb{P}_{\max}} \models [\langle H(\omega_2), \in, \mathrm{NS}_{\omega_1}, A \rangle \models \varphi] \, .$$

Forcing with $\mathbb{P}_{\max}$ does not add reals, so there is no need to reinterpret A in the last line of the theorem. The theorem says that any such Π_2 statement that we can force in any extension must hold in the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$, so $H(\omega_2)$

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of $L(\mathbb{R})^{\mathbb{P}_{\max}}$ is maximal, or complete, in a certain sense; among other things, it models ZFC, Martin's Axiom, certain fragments of Martin's Maximum [1], and the negation of the Continuum Hypothesis. The reinterpretation A^* will be defined later, in terms of tree representations for sets of reals. We will not give the definition of Woodin cardinals (but see [4]).

We have reworked the standard proof of Theorem 1 in order to minimize the prerequisites. In particular, the need for (mentioning) sharps has been eliminated. However, they and much more will need to be reintroduced to go any further than the material presented here.

"Woodin's book on $\mathbb{P}_{\max}$, *The axiom of determinacy, forcing axioms and the nonstationary ideal* [9] runs to around 1000 pages. My article for the Handbook of Set Theory [5], introducing $\mathbb{P}_{\max}$, has about 65. The advance notes for these lectures are about 30 pages, and previous lecture courses have taken about 12-15 hours to cover $\mathbb{P}_{\max}$; so today will, of course, have to be brief..."

2 Setup: iterations and the definition of $\mathbb{P}_{\max}$

Suppose that $M \models$ ZFC and than $I \in M$ is a normal ideal on ω_1^M in M. Force over M with $((\mathcal{P}(\omega_1) \setminus I)/I)^M$. The resulting generic G is now an M-normal ultrafilter on ${\omega_1}^M$; so we may form the corresponding ultrapower and elementary embedding $j : M \to Ult(M, G) := \{f : {\omega_1}^M \to M \mid f \in M\}/ =_G$. ("We'll use this a thousand times today.") Note that $\mathrm{crit}(j) = {\omega_1}^M$, $j({\omega_1}^M) \geq {\omega_2}^M$, $\mathrm{Ord}^{Ult(M,G)} = \mathrm{Ord}^M$, and for $A \in \mathcal{P}(\omega_1)^M$, $A \in G \leftrightarrow {\omega_1}^M \in j(A)$. There is no need to assume that A is transitive, though it will be in the cases were are interested in. When an ultrapower is well-founded, we identify it with its transitive collapse.

Definition 2. *I is* precipitous *if $Ult(M, G)$ thus constructed is well-founded from the point of view of $M[G]$, for all M-generic G. (N.B. this is definable in M via forcing.)*

We need a pair of theories satisfying the following conditions.

- T_0, a theory consistent with ZFC and strong enough to make sense of the generic ultrapower construction above and prove that $j : M \to Ult(M, G)$ is elementary.
- T_1, a theory consistent with ZFC and at least as strong as T_0 + "every set lies in some $H(\kappa) \models T_0$."

In [5], we take T_0 (which we call ZFC$^\circ$) to be ZFC - Replacement - Powerset plus "$\mathcal{P}(\mathcal{P}(\omega_1))$ exists" plus the scheme saying that definable trees of height ω_1 have maximal branches. Then we let $T_1 = T_0$ + Powerset + Choice + Σ_1-Replacement (though we don't express it in these terms). In [9], Woodin has an even weaker fragment of ZFC (which he calls ZFC*) playing the role of T_0. Today we may as well let T_0 be ZFC and T_1 be ZFC plus the existence of a proper class of strongly inaccessible cardinals. ¿From now on we will just use the terms T_0 and T_1.

Our basic construction is the generic ultrapower. We now extend to iterated ultrapowers. Suppose we have (M_0, I_0), $G_0 \subseteq (\mathcal{P}(\omega_1)/I_0)^{M_0}$, $j_0 : (M_0, I_0) \to Ult(M_0, G_0)$, all as before; let $M_1 = Ult(M_0, G_0)$, $I_1 = j_0(I_0)$. Now we can take the generic ultrapower of M_1 by I_1, and iterate. At limit stages, we have a directed system of elementary embeddings, so can just take the direct limit, so we can keep going up to length ω_1. (No further, as if we force again there, we collapse ${\omega_1}^V$, so are back to countable length!) Note that the final model of the iteration, M_{ω_1}, is an element of $H(\omega_2)$.

Definition 3. *An* iteration *of (M, I) (as above; M countable) of length γ consists of M_α, I_α $(\alpha \leq \gamma)$, G_η $(\eta < \gamma)$, and $j_{\alpha,\beta}$ $(\alpha \leq \beta \leq \gamma)$, satisfying*

- $M_0 = M$, $I_0 = I$
- G_η *is M_η-generic for* $(P(\omega_1)/I_\eta)^{M_\eta}$
- $j_{\eta,\eta+1}$ *is the canonical embedding of M_η into* $Ult(M_\eta, G_\eta) = M_{\eta+1}$
- $j_{\alpha,\beta} : M_\alpha \to M_\beta$ *are a commuting family of elementary embeddings*
- $I_\beta = j_{0,\beta}(I)$
- *For limit β, M_β is the direct limit of $\{M_\alpha \mid \alpha < \beta\}$ under the embeddings $j_{\alpha,\eta}$ ($\alpha \leq \eta < \beta$).*

In practice, we almost always have $\gamma = {\omega_1}^N$ for some larger $N \supseteq M$. We will generally write $\langle M_\alpha, I_\alpha, G_\eta, j_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma,\ \eta < \gamma\rangle$ for the iteration, or just "j is an iteration" to mean that j is the $j_{0,\gamma}$ of an iteration, with $j(M)$ for M_γ. (As we will see, in some circumstances, if we know M_0, M_γ, $j_{0,\gamma}$, we can (with slight assumptions) recover the full iteration.) We say that the M_α's are *iterates* of (M, I); (M, I) is *iterable* if all iterates are well-founded; and (M, I) is an *iterable pair* if M is a countable transitive model of T_0, I a normal ideal on $\mathcal{P}(\omega_1)$ in M, and (M, I) is iterable.

If M is well-founded and $M \models$ "I is precipitous," then certainly (M, I) is finitely iterable (i.e., its finite-length iterations produce wellfounded models); and in fact, we will show that in this case (M, I) is iterable to any $\alpha \in \mathrm{Ord}^M$.

The proof of the following lemma is left an exercise (the proof is by induction on the length of the iteration). In a typical application, M is $H(\kappa)^N$ for some suitable κ.

Lemma 4. *Suppose that $M \in N$ are models of T_0, M is closed under ω_1-sequences from N, and $\mathcal{P}(\mathcal{P}(\omega_1))^M = \mathcal{P}(\mathcal{P}(\omega_1))^N$. Let $I \in M$ be an M-normal ideal on ${\omega_1}^M$. Then the following hold.*

- *for each iteration* $\langle M_\alpha, I_\alpha, G_\eta, j_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma,\ \eta < \gamma\rangle$ *of* (M, I) *there is a unique iteration* $\langle N_\alpha, I_\alpha, G_\eta, j^*_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma,\ \eta < \gamma\rangle$ *of* (N, I) *such that* $\forall\beta \leq \gamma$, $j^*_{0,\beta}(M) = M_\beta$, M_β *is closed under* ω_1*-sequences from* N_β, $\mathcal{P}(\mathcal{P}(\omega_1))^{M_\beta} = \mathcal{P}(\mathcal{P}(\omega_1))^{N_\beta}$, *and* $j^*_{\alpha,\beta}\restriction_{M_\alpha} = j_{\alpha,\beta}$.

- *for each iteration* $\langle N_\alpha, I_\alpha, G_\eta, j^*_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma,\ \eta < \gamma\rangle$ *of* (N, I) *there is a unique iteration* $\langle M_\alpha, I_\alpha, G_\eta, j_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma,\ \eta < \gamma\rangle$ *of* (M, I) *such that* $\forall\beta \leq \gamma$, $j^*_{0,\beta}(M) = M_\beta$, M_β *is closed under* ω_1*-sequences from* N_β, $\mathcal{P}(\mathcal{P}(\omega_1))^{M_\beta} = \mathcal{P}(\mathcal{P}(\omega_1))^{N_\beta}$, *and* $j^*_{\alpha,\beta}\restriction_{M_\alpha} = j_{\alpha,\beta}$.

Corollary 5. *In the context of Lemma 4, if* (M, I) *has an ill-founded iterate by an iteration of length* α*, then so does* (N, I).

Lemma 6 below then shows that (M, I) is iterable if N contains ω_1 (recall that iterations can have length at most ω_1, and note that an illfounded iteration of length ω_1 must be illfounded at some countable stage).

First we fix a coding of elements of $H(\omega_1)$ by reals. Fix a recursive bijection $\pi : \omega \times \omega \to \omega$, and say $X \subseteq \omega$ *codes* $a \in H(\omega_1)$ if

$$(\mathrm{tc}(\{a\}), \in) \cong (\omega, \{(i, j) \mid \pi(i, j) \in X\}),$$

where $tc(b)$ denotes the transitive closure of b. Then $\in$ and $=$ are Σ^1_1 (as permutations of ω induce different codes for the same object).

Lemma 6. *Suppose that* N *is a transitive model of* T_1, $\gamma \in \mathrm{Ord}^N$*, and* I *is a normal precipitous ideal on* ${\omega_1}^N$ *in* N*. Then any iterate of* (N, I) *by an iteration of length* γ *is well-founded.*

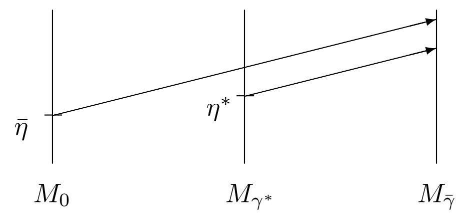
Proof. It suffices to prove that iterations of the form $(H(\kappa)^N, I)$ produce well-founded models for all $\kappa \in N$ such that $H(\kappa)^N \models T_0$; for if any iterate of N is ill-founded, then some ordinal in N is large enough to witness this (i.e. $\sup(\mathrm{rge}(f))$, where f witnesses ill-foundedness) and by assumption (as $N \models T_1$) this is contained in some $H(\kappa)^N$ that models T_0.

Let $(\bar\gamma, \bar\kappa, \bar\eta)$ be the lexicographically minimal triple (γ, κ, η) satisfying (with N) the formula $\varphi(N, \gamma, \kappa, \eta)$ defined by "$H(\kappa)^N \models T_0$ and there exists an iteration of $(H(\kappa)^N, I)$ of length γ which is ill-founded below the image of η".

Using our fixed coding of elements of $H(\omega_1)$ by reals there is a Σ^1_1 formula $\varphi'(x, y, z)$ saying "x codes a model of T_0 and a normal ideal in the model on the ω_1 of the model and there exists an iteration of this pair whose length is coded by y and which is illfounded below the image of the element of this model coded by z."

For all cardinals $\kappa, \rho \in N$ and all ordinals $\gamma, \eta \in N$, if $\rho \in N$ is larger than $|H(\kappa)|^N$, $|\eta|^N$ and $|\gamma|^N$, then there exist reals coding $H(\kappa)^N$, η, and γ in any forcing extension of N by $\mathrm{Coll}(\omega, \rho)$. Such an extension is correct about whether these reals satisfy φ'. However, this is a homogeneous forcing extension of N; so there is a formula $\psi(\gamma, \kappa, \eta)$ saying that in every forcing extension in which $H(\kappa)$ (of the ground model), η and γ are all countable there exist reals coding $H(\kappa)$, η and γ which satisfy φ'. It follows that that $N \models \psi(\gamma, \kappa, \eta)$ if and only if $\varphi(N, \gamma, \kappa, \eta)$ holds, and furthermore, for all well-founded iterates N^* of N, and all $\gamma, \kappa, \eta \in N^*$, $N^* \models \psi(\gamma, \kappa, \eta)$ if and only if $\varphi(N^*, \gamma, \kappa, \eta)$ holds.

Since I is precipitous in N, $\bar{\gamma}$ is a limit ordinal, and clearly $\bar{\eta}$ is a limit ordinal as well. Fix an iteration $\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \bar{\gamma}\rangle$ of $(H(\bar{\kappa})^N, I)$ such that $j_{0\bar{\gamma}}(\bar{\eta})$ is not wellfounded, and let $\langle N_\alpha, G_\beta, j'_{\alpha\delta} : \beta < \alpha \leq \delta \leq \bar{\gamma}\rangle$ be the corresponding iteration of N as in Lemma 4. By the minimality of $\bar{\gamma}$ we have that N_α is wellfounded for all $\alpha < \bar{\gamma}$. Since $M_{\bar{\gamma}}$ is the direct limit of the iteration leading up to it, we may fix $\gamma^* < \bar{\gamma}$ and $\eta^* < j_{0\gamma^*}(\bar{\eta})$ such that $j_{\gamma^*\bar{\gamma}}(\eta^*)$ is not wellfounded. By Lemma 4, $j'_{\gamma^*,\bar{\gamma}}(\eta^*) = j_{\gamma^*,\bar{\gamma}}(\eta^*)$ and $j'_{\gamma^*,\bar{\gamma}}(\bar{\eta}) = j_{\gamma^*,\bar{\gamma}}(\bar{\eta})$.



But now, $N_{\gamma^*} \models \psi(\bar{\gamma} - \gamma^*, j_{0,\gamma^*}(\bar{\kappa}), \eta^*)$, $\bar{\gamma} - \gamma^* \leq \bar{\gamma}$, and $\eta^* < j_{0,\gamma^*}(\bar{\eta})$, contradicting minimality of $(j_{0,\gamma^*}(\bar{\gamma}), j_{0,\gamma^*}((\bar{\kappa}), j_{0,\gamma^*}(\bar{\eta}))$ in N_{γ^*}. $\square$

We note that ZFC does not imply the existence of iterable pairs. However, by Lemma 6, if there exist a normal, precipitous ideal J on ω_1, and a measurable cardinal κ with a κ-complete ultrafilter μ, then there exist iterable pairs. The main point here is that if $\theta > \kappa$ is a regular cardinal and X is a countable elementary submodel of $H(\theta)$ with $\kappa, J \in X$, then X can be end-extended below κ by taking γ to be any member of $\bigcap(X \cap \mu)$, and letting $X[\gamma]$ be the set of values $f(\gamma)$ for all functions f in X with domain κ. Applying this fact ω_1 many times, we get that the transitive collapse M of $X \cap V_\kappa$ is a countable model which is a rank initial segment of a model containing ω_1. Letting I be the image of J under the transitive collapse, then, (M, I) is an iterable pair. This is a special case of the proof of Lemma 22, and a key point in Woodin's proof (which appears in Chapter 3 of [9]) that if there exists a measurable cardinal and the nonstationary ideal on ω_1 is saturated, then CH fails.

If there is a precipitous ideal on ω_1, then sharps exist for subsets of ω_1, and a countable iterable model will be correct about these sharps. We will work around this today to avoid having to talk about sharps.

Lemma 7. *If (M, I) is an iterable pair and A is an element of $\mathcal{P}(\omega_1)^M$, then $(\omega_1^{L[A]})^M = \omega_1^{L[A]}$.*

Proof. Let $\langle M_\alpha, I_\alpha, G_\eta, j_{\alpha,\beta} \mid \alpha \leq \beta \leq \omega_1,\ \eta < \omega_1\rangle$ be an iteration of (M, I). The ordinals of M and M_1 are the same, so $L[A]^M = L[A]^{M_1}$. The critical point of $j_{1\omega_1}$ is greater than ω_1^M, and thus greater than the ω_1 of $L[A]^M$. The restriction of $j_{1\omega_1}$ to $L[A]^M$ embeds $L[A]^M$ elementarily into $L[A]^{M_{\omega_1}}$, which means that $L[A]^M$ and $L[A]^{M_{\omega_1}}$ have the same ω_1. Since $\omega_1 \subset M_{\omega_1}$, $(\omega_1^{L[A]})^{M_{\omega_1}} = \omega_1^{L[A]}$. $\square$

Now we can define $\mathbb{P}_{\max}$.

Definition 8. *The partial order $\mathbb{P}_{\max}$ is the set of pairs $\langle (M, I), a\rangle$ such that*

1. *M is a countable transitive model of T_0 + $\mathrm{MA}_{\aleph_1}$*
2. *(M, I) is an iterable pair*
3. *$a \in \mathcal{P}(\omega_1)^M$ and $\exists x \in \mathcal{P}(\omega)^M$ such that $\omega_1^{L[x,a]} = \omega_1^M$*

ordered by: $p < q$ (where $p = \langle (M, I), a\rangle$, $q = \langle (N, J), b\rangle$) if there is some iteration $j : (N, J) \to (N^*, J^*)$ such that

1. *$j \in M$*
2. *$j(b) = a$*
3. *$J^* = N^* \cap I$ (and hence $j(\omega_1^N) = \omega_1^M$)*
4. *$q \in H(\omega_1)^M$*

Note that since $j \in M$ in definition of the $\mathbb{P}_{\max}$ order above, N and N^* are both in M as well.

Definition 9. *We say that (M, I) is a $\mathbb{P}_{\max}$* precondition *if there exists an a such that $\langle (M, I), a\rangle \in \mathbb{P}_{\max}$, or equivalently just if (M, I) satisfies conditions 1 and 2 in the definition of $\mathbb{P}_{\max}$ conditions above.*

Suppose that (M, I) is an iterable pair, and $j\colon (M, I) \to (M', I')$ is an iteration. Then for any $A \in \mathcal{P}(\omega_1)^M$ which is bounded in ω_1^M, $j(A) = A$. By Lemma 7, it follows then that $\omega_1^{L[A]} < \omega_1^M$, since $j(\omega_1^M) > \omega_1^M$ if j is nontrivial. Therefore, the set a from a $\mathbb{P}_{\max}$ condition $\langle (M, I), a\rangle$ must always be unbounded in ω_1^M to make ${\omega_1}^{L[x,a]} = {\omega_1}^M$ possible.

If $p_0 < p_1 < p_2$ ($p_i = \langle (M_i, I_i), a_i\rangle$), and these are witnessed by $j_{1,0}$, $j_{2,1}$, then $p_0 < p_2$ is witnessed by $j_{1,0}(j_{2,1})$: $j_{1,0} \in H(\omega_2)^{M_0}$, $j_{2,1} \in H(\omega_2)^{M_1}$; $j_{2,1}$ is an iteration of (M_2, I_2), and $j_{1,0}((M_2, I_2)) = (M_2, I_2)$.

Under our fixed coding, "(M, I) is iterable" is Π^1_2 in a code for (M, I) : roughly, "for anything satisfying the first-order properties of being an iteration, either there is no infinite descending sequence in the ordinals of the final model, or there is an infinite descending sequence in the indices of the iteration." Since iterable models embed elementarily into models containing ω_1, they are $\mathbf{\Pi}^1_2$-correct. It follows that "(M, I) is iterable" is absolute to iterable models containing a code for (M, I).

So now we see that $\mathbb{P}_{\max} \in L(\mathbb{R})$ — all constructions involved are nicely codable.

3 First properties of $\mathbb{P}_{\max}$

The requirement that the models in $\mathbb{P}_{\max}$ conditions satisfy $\mathrm{MA}_{\aleph_1}$ is used for a particular consequence of $\mathrm{MA}_{\aleph_1}$ known as *almost disjoint coding* [2]. That is, it follows from $\mathrm{MA}_{\aleph_1}$ that if $Z = \{z_\alpha : \alpha < \omega_1\}$ is a collection of infinite subsets of ω whose pairwise intersections are finite (i.e., Z is an *almost disjoint family*), then for each $B \subseteq \omega_1$ there exists a $y \subseteq \omega$ such that for all $\alpha < \omega_1$, $\alpha \in B$ if and only if $y \cap z_\alpha$ is infinite. This is used to show that if $\langle (M, I), a\rangle$ is a $\mathbb{P}_{\max}$ condition, then any iteration of (M, I) is uniquely determined by the image of a (Lemma 10 below), so the order on each comparable pair of conditions is witnessed by a unique iteration.

Lemma 10. *Let $\langle (M, I), a\rangle$ be a $\mathbb{P}_{\max}$ condition and let A be a subset of ω_1. Then there is at most one iteration of (M, I) for which A is the image of a.*

Proof. Fix a real x in M such that $\omega_1^M = \omega_1^{L[a,x]}$, and let $Z = \langle z_\alpha : \alpha < \omega_1^M\rangle$ be the almost disjoint family defined recursively from the constructibility order in $L[a, x]$ on $\mathcal{P}(\omega)^{L[a,x]}$ (using a and x as parameters) by letting $\langle z_i : i < \omega\rangle$ be the constructibly least partition of ω into infinite pieces, and, for each $\alpha \in [\omega, \omega_1^M)$, letting z_α be the constructibly least infinite $z \subset \omega$ almost disjoint from each z_β $(\beta < \alpha)$. Suppose that

$$\mathcal{I} = \langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \gamma\rangle$$

and

$$\mathcal{I}' = \langle M'_\alpha, G'_\beta, j'_{\alpha\delta} : \beta < \alpha \leq \delta \leq \gamma'\rangle$$

are two iterations of (M, I) such that $j_{0\gamma}(a) = A = j'_{0\gamma'}(a)$. Then

$$j_{0\gamma}(\omega_1^M) = \omega_1^{L[x,A]} = j_{0\gamma}(\omega_1^{M'})$$

and $j_{0\gamma}(Z) = j'_{0\gamma'}(Z)$ (this uses Lemma 7 to see that the constructibility order on reals in $L[A, x]$ is computed correctly in M_γ and $M'_{\gamma'}$). Let $\langle z_\alpha : \alpha < j_{0\gamma}(\omega_1^M)\rangle$ enumerate $j_{0\gamma}(Z)$.

Without loss of generality, $\gamma \leq \gamma'$. We show by induction on $\alpha < \gamma$ that, for each such α, $G_\alpha = G'_\alpha$. This will suffice. Fix α and suppose that

$$\{G_\beta : \beta < \alpha\} = \{G'_\beta : \beta < \alpha\}.$$

Then $M_\alpha = M'_\alpha$. For each $B \in \mathcal{P}(\omega_1)^{M_\alpha}$, $B \in G_\alpha$ if and only if $\omega_1^{M_\alpha} \in j_{\alpha(\alpha+1)}(B)$, and $B \in G'_\alpha$ if and only if $\omega_1^{M_\alpha} \in j'_{\alpha(\alpha+1)}(B)$. Applying almost disjoint coding, fix $x \in \mathcal{P}(\omega)^{M_\alpha}$ such that for all $\eta < \omega_1^{M_\alpha}$, $\eta \in B$ if and only if $x \cap z_\eta$ is infinite. Then $B \in G_\alpha$ if and only if $x \cap z_{\omega_1^{M_\alpha}}$ is infinite if and only if $B \in G'_\alpha$. $\square$

Lemma 11. *(T_0) Suppose that (M, I) is an iterable pair, and J is a normal ideal on ω_1. Then there exists an iteration $j\colon (M, I) \to (M^*, I^*)$ of length ω_1 such that $I^* = M^* \cap J$.*

Proof. Note that $I^* \subseteq M^* \cap J$ holds for any such ω_1-length iteration. To see this, first note that the critical sequence of an iteration of length ω_1 is a club. Every element B of I^* is $j_{\alpha,\omega_1}(b)$ for some $\alpha < \omega_1$ and $b \in I_\alpha$. Then for all $\beta \in [\alpha, \omega_1)$, $j_{\alpha,\beta}(b) \notin G_\beta$, so ${\omega_1}^{M_\beta} \notin j_{\alpha,\omega_1}(b) = B$; thus $B \in \mathrm{NS}_{\omega_1}$, but J is normal, so $NS_{\omega_1} \subseteq J$.

Conversely, for $\supseteq$: as J is normal, we may let $\langle E_i^\alpha \mid \alpha < \omega_1, i < \omega \rangle$ be a partition of ω_1 into J-positive pieces. Now, as we construct an iteration, let $\{e_i^\alpha \mid i < \omega\}$ enumerate $\mathcal{P}(\omega_1)^{M_\alpha} \setminus I_\alpha$, and build each G_β in such a way that if ${\omega_1}^{N_\beta} \in E_i^\alpha$ for some $\alpha \leq \beta$ and $i < \omega$, then $j_{\alpha,\beta}(e_i^\alpha)$ is in G_β.

Now, for all $B \in \mathcal{P}(\omega_1)^{M_\alpha} \setminus I_{\omega_1}$, $\exists \alpha < \omega_1$, $i < \omega$ such that $B = j_{\alpha,\omega_1}(e_i^\alpha)$ and for all $\beta \in [\alpha, \omega_1)$, ${\omega_1}^{M_\beta} \in E_i^\alpha \Rightarrow {\omega_1}^{M_\beta} \in j_{0,\beta+1}(e_i^\alpha) \Rightarrow {\omega_1}^{M_\beta} \in B$.

So in particular, we have a club $C \subseteq \omega_1$ such that $C \cap E_i^\alpha \subseteq B$, so $B \notin J$. $\square$

We may consider this as an *iteration game* $G((M, I), J, B)$: two players collaborate on building an iteration of (M, I), and play is as follows at each round α:

- if $\alpha \notin B$, player I does nothing, and player II chooses G_α;
- if $\alpha \in B$, player I specifies some element for G_α, and player II must choose some G_α containing it.

Player I wins if $I_{\omega_1} = M_{\omega_1} \cap J$. The above proof shows that player I has a winning strategy iff $B \notin J$. (More precisely, it shows $\Leftarrow$; $\Rightarrow$ is because if $B \in J$ then II may choose some I-positive set and keep its images out of every G_α.)

The following lemma shows that $\mathbb{P}_{\max}$ satisfies a homogeneity property strong enough to imply that the theory of the generic extension can be computed in the ground model. Since the existence of a proper class of Woodin cardinals implies that the theory of $L(\mathbb{R})$ is generically absolute, it implies that the theory of the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$ is generically absolute as well.

Lemma 12. *Suppose for each $x \in H(\omega_1)$ there exists a $\mathbb{P}_{\max}$ precondition (M, I) such that $x \in M$. Then $\forall p_0, p_1 \in \mathbb{P}_{\max}$, $\exists q_0, q_1 \in \mathbb{P}_{\max}$ such that each $q_i \leq p_i$, and $\mathbb{P}_{\max} \restriction q_0 \cong \mathbb{P}_{\max} \restriction q_1$.*

Proof. Take $p_i = \langle (M_i, I_i), a_i \rangle$. Then let (N, J) be a $\mathbb{P}_{\max}$ precondition such that $p_0, p_1 \in H(\omega_1)^N$. Now take $j_i \colon (M_i, I_i) \to (M_i^*, I_i^*)$ to be iterations in N such that $I_i^* = M_i^* \cap J$ (we may do so, by the previous theorem applied in N).

Now set $q_i = \langle (N, J), j_i(a_i) \rangle \in \mathbb{P}_{\max}$. Certainly these satisfy $q_i < p_i$ as desired. (To see that these q_i are indeed conditions, note that the witnessing x_i for p_i (i.e., the $x \in \mathcal{P}(\omega)^{M_i}$ such that $\omega_1^{L[x,a_i]} = \omega_1^{M_i}$) still works for q_i.) But now $\mathbb{P}_{\max} \restriction q_0 \cong \mathbb{P}_{\max} \restriction q_1$, for given any $r_0 = \langle (N', I'), b \rangle < q_0$, there is unique $j : (N, J) \to (N^*, J^*)$ witnessing this (and we have $b = j(j_0(a_0))$); now take r_0 to $r_1 := \langle (N', J'), j(j_1(a_1)) \rangle < q_1$, also witnessed by j. $\square$

Given $\gamma \in [\omega_1, \omega_2)$, a *canonical function* for γ is a function $f \colon \omega_1 \to \omega_1$ such that for some (equivalently, every) bijection $\pi : \omega_1 \to \gamma$, $\{\alpha < \omega_1 \mid \mathrm{ot}(\pi[\alpha]) =$

$f(\alpha)\}$ contains a club. In a normal ultrapower context, we then have: $f\colon \omega_1 \to \mathrm{Ord}$, $[f]_{Ult} = j(f)(\omega_1) = \mathrm{ot}(\pi[\omega_1]) = \gamma$.

Suppose that $\langle M_\alpha, I_\alpha, G_\eta, j_{\alpha,\beta} \mid \alpha \leq \beta \leq \omega_1,\ \eta < \omega_1\rangle$ is an iteration of length ω_1 of some (M, I), and let $\pi : \omega_1 \to \mathrm{Ord}^{M_{\omega_1}}$ be a bijection; then π induces a canonical function g. For club-many $\alpha < \omega_1$, ${\omega_1}^{M_\alpha} = \alpha$, and also $\pi[\alpha] = j_{\alpha,\omega_1}[\mathrm{Ord}^{M_\alpha}]$ (recall that we take direct limits at limit stages of iterations); so there is a club $C \subset \omega_1$ such that for all $\alpha \in C$, $\mathrm{ot}(\pi[\alpha]) = \mathrm{Ord}^{M_\alpha}$. If $f \in ({\omega_1}^{\omega_1})^{M_\beta}$, for some $\beta < \alpha$, then for any $\alpha \in [\beta, \omega_1)$,

$$j_{\beta,\alpha+1}(f)({\omega_1}^{M_\alpha}) < {\omega_1}^{M_{\alpha+1}} < \mathrm{Ord}^{M_{\alpha+1}} = \mathrm{Ord}^{M_\alpha},$$

which equals $g(\alpha)$ if $\alpha \in C$. It follows that for any $f \in ({\omega_1}^{\omega_1})^{M_{\omega_1}}$, $g(\alpha) > f(\alpha)$ for club-many α. We will use this fact to show that $\mathbb{P}_{\max}$ σ-closed (this is an alternate proof avoiding sharps; some of what follows can be done more easily and in more generality with sharps).

Lemma 13. *Suppose that for each $x \in H(\omega_1)$ there exists a $\mathbb{P}_{\max}$ precondition (M, I) such that $x \in M$. If $p_i \in \mathbb{P}_{\max}$ $(i < \omega)$ are such that $\forall i\ p_{i+1} < p_i$, then $\exists q \in \mathbb{P}_{\max}$ such that $\forall i\ q < p_i$.*

The proof of Lemma 13 involves some new notions. Say that $p_i = \langle (M_i, I_i), a_i\rangle$, and for each $j < i < \omega$, let $k_{i,j} : M_j \to M_j^* \in M_i$ be the unique witness for $p_i < p_j$. By the uniqueness of witnesses, the $k_{i,j}'s$ commute, so let N_i, J_i be the images of M_i, I_i in the limit of the directed system given by the embeddings $k_{i,j}$. Each (N_i, J_i) is an iterate of the corresponding (M_i, I_i) by an iteration of length $\sup\{\omega_1^{M_i} : i < \omega\}$, so each N_i is wellfounded. Let $b = \bigcup_{i<\omega} a_i$. Then:

1. Each (N_i, J_i) is iterable;
2. for all i, ${\omega_1}^{N_i} = \sup_{j<\omega} {\omega_1}^{M_j}$;
3. $i < j \Rightarrow N_i \in H(\omega_2)^{N_j}$;
4. for all i, $J_i = J_{i+1} \cap N_i$;
5. for each i there exists some iteration $j_i : (M_i, I_i) \to (N_i, J_i)$ in N_{i+1} such that $j_1(a_i) = b$ (and so in N_{i+1}, there is a canonical function for Ord^{N_i} that dominates on a club every member of $(\omega_1^{\omega_1})^{N_i}$).

We call a sequence $\langle (N_i, J_i) : i < \omega\rangle$ satisfying (1) to (5) above a $\mathbb{P}_{\max}$ *limit sequence*. An $\langle (N_i, J_i) \mid i < \omega\rangle$*-normal ultrafilter* is a filter $G \subseteq \bigcup_i (\mathcal{P}(\omega_1)^{N_i} \setminus J_i)$ such that for all $i < \omega$, and for all regressive $f \in ({\omega_1}^{\omega_1})^{N_i}$ $\exists e \in G$ such that f is constant on e. Then we have $Ult(\langle (N_i, J_i) : i < \omega\rangle, G)$, i.e. a sequence of models whose jth model $\left[Ult(\langle (N_i, J_i) : i < \omega\rangle, G)\right]_j$ is

$$\{f : {\omega_1}^{N_0} \to N_j \mid f \in \bigcup\{N_i \mid i < \omega\}\}/=_G .$$

Now we will iterate this operation.

Definition 14. *An iteration of $\langle (N_i, J_i) : i < \omega\rangle$ of length γ is some*

$$\langle\langle (N_i^\alpha, J_i^\alpha) : i < \omega\rangle, G_\eta, j_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma, \eta < \gamma\rangle$$

in which:

- *each G_η is a langle$(N_i^\eta, J_i^\eta) \mid i < \omega\rangle$-normal ultrafilter contained in $\bigcup_i (\mathcal{P}(\omega_1)^{N_i^\eta} \setminus J_i^\eta)$ such that for each $i < \omega$, $j_{\eta,\eta+1} \upharpoonright N_i^\eta \to N_i^{\eta+1}$ is the induced ultrapower*

- *the $j_{\alpha,\beta}$ commute, and for limit β, N_i^β is the direct limit of N_i^α $(\alpha < \beta)$ under $j_{\alpha,\rho} \upharpoonright N_i^\alpha$ $(\alpha \leq \rho < \beta)$.*

In an iteration of this form, for each pair i, α, there is in N_{i+1}^α an iteration of (M_i, I_i) of length ${\omega_1}^{N_0^\alpha}$, with final model N_i^α. Since each (M_i, I_i) is iterable, the wellfoundedness of each N_i^α will follow from the wellfoundedness of $\omega_1^{N_0^\alpha}$.

For each $\alpha < \gamma$,

$$\omega_1^{N_0^{\alpha+1}} = \sup\{\mathrm{Ord}^{N_i^\alpha} \mid i < \omega\}.$$

To see this, fix for each $i \in \omega$ a canonical function $g_i \in N_{i+1}^\alpha$ for Ord^{N_i}. Then each g_i dominates on a club every member of $(\omega_1^{\omega_1})^{N_i^\alpha}$. The g_i are cofinal under mod-NS_{ω_1} domination in $\bigcup_i ({\omega_1}^{\omega_1})^{N_i^\alpha}$, and each g_i represents an ordinal in N_{i+1}^α in this ultrapower, which shows that $\omega_1^{N_0^{\alpha+1}} = \sup\{\mathrm{Ord}^{N_i^\alpha} \mid i < \omega\}$. For limit β,

$${\omega_1}^{N_0^\beta} = \sup\{{\omega_1}^{N_0^\alpha} \mid \alpha < \beta\}.$$

It follows that each N_0^α is wellfounded.

We have shown the following.

Fact 15. *All iterations of $\mathbb{P}_{\max}$ pre-limit sequences give well-founded models.*

Again this can be rephrased in terms of games. Let $G_\omega(\langle (N_i, J_i) \mid i < \omega\rangle, I, B)$ be the game of length ω_1, in which I and II collaborate to build an iteration of $\langle (N_i, J_i) : i < \omega\rangle$ of length ω_1, in which at stage α:

- if $\alpha \in B$, player I chooses $e \in \bigcup_i (\mathcal{P}(\omega_1)^{N_i^\alpha} \setminus J_i^\alpha)$, and player II chooses G_α, a $\langle (N_i^\alpha, J_\alpha^i) \mid i < \omega\rangle$-normal ultrafilter contained in $\bigcup_i (\mathcal{P}(\omega_1)^{N_i^\alpha} \setminus J_i^\alpha)$, with $e \in G_\alpha$;

- if $\alpha \notin B$, player I does nothing, and player II chooses any suitable G_α.

Player I wins if $\forall i \; j_{0,\omega_1}(J_i) = I \cap M_i^{\omega_1}$. The argument just given (along with the argument for Lemma 11) shows the following.

Lemma 16. *Suppose $\langle (N_i, J_i) \mid i < \omega\rangle$ is a $\mathbb{P}_{\max}$ pre-limit-sequence, I is a maximal ideal on ω_1 and $B \subseteq \omega_1$. Then Player I has a winning strategy in $G_\omega(\langle (N_i, J_i) \mid i < \omega\rangle, I, B)$ if and only if B is not in I.*

We now return to the proof of Lemma 13. We have that the limit sequence $\langle (N_i, J_i) : i < \omega \rangle$ induced by the descending sequence p_i $(i \in \omega)$ is iterable. Fix a $\mathbb{P}_{\max}$ precondition (M', I') such that this sequence is in $H(\omega_1)^{M'}$. Apply a winning strategy for player I in M' for $G_{\omega_1}(\langle (N_i, J_i) \mid i < \omega \rangle, I', \omega_1)$ to get an iteration j of $\langle (N_i, J_i) : i < \omega \rangle$ of length $\omega_1^{M'}$. Then for all $i < \omega$, $j(j_i)$ witnesses that $p_i > \langle (M', I'), j(b) \rangle$.

Thus $\mathbb{P}_{\max}$ forcing is σ-closed, so it does not add any reals; so $L(\mathbb{R})^{V^{\mathbb{P}_{\max}}} = L(\mathbb{R})^V$.

4 Existence of $\mathbb{P}_{\max}$ conditions

Definition 17. *Given $A \subseteq \mathbb{R}$, and an iterable pair (M, I), we say (N, I) is A-*iterable *if $A \cap M \in M$ and for any iteration $j : (M, I) \to (M^*, I^*)$, $j(A \cap M) = A \cap M^*$.*

In this section we will work through a proof of the following existence theorem for $\mathbb{P}_{\max}$ conditions.

Lemma 18 (Main existence lemma)**.** *Suppose there are infinitely many Woodin cardinals below some measurable cardinal, and let $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$. Then there exists an A-iterable $\mathbb{P}_{\max}$ precondition (M, I) such that for every set forcing extension M^+ of M and every precipitous ideal $I^+ \in M^+$ on $\omega_1^{M^+}$, (M^+, I^+) is A-iterable.*

We need to introduce towers of measures and homogeneous tree. See [8] for a detailed discussion of this material

Definition 19. *Given $Z \neq \phi$, a* tower of measures *on Z is a sequence $\langle \mu_i \mid i < \omega \rangle$ such that each $\mu_i \subseteq \mathcal{P}(Z^i)$ is an ultrafilter, and for all $k < i < j$ and all $A \in \mu_i$, we have $\{b \in Z^j \mid b \restriction i \in A\} \in \mu_j$ and $\{b \restriction k \mid b \in A\} \in \mu_k$.*

Such a tower is countably complete *if whenever $\langle A_i \mid i < \omega \rangle$ is such that eaach $A_i \in \mu_i$, there is $a \in Z^\omega$ such that $\forall i$ $a \restriction i \in A_i$.*

We note briefly that countable completeness is equivalent to: the direct limit of $Ult(V, \mu_i)$ is well-founded.

Definition 20. *A* tree *on $\omega \times Z$ is a set $T \subseteq (\omega \times Z)^{<\omega}$ such that for all $i < \omega, t \in T$ we have $t \restriction i \in T$. The projection of T is $p[T] := \{y \in \omega^\omega \mid \exists c \in Z^\omega \ \forall i < \omega \ (y \restriction i, c \restriction i) \in T\}$.*

Such a tree is weakly κ-homogeneous *(for κ a cardinal) if there exist κ-complete ultrafilters $\mu_{a,b} \subseteq \mathcal{P}(Z^{|a|})$ such that $\forall a, b \in \omega^{<\omega}$ with $|a| = |b|$,*

$$\{c \in Z^{|a|} \mid (a, c) \in T\} \in \mu_{a,b},$$

and such that for each $x \in p[T]$ there exists a $b \in \omega^\omega$ such that $\langle \mu_{x \restriction i, b \restriction i} \mid i < \omega \}$ is a countably complete tower.

Weakly homogeneous trees originated from work of Kechris, Martin and Solovay. The following fact is due to Woodin. A proof appears in [6].

Fact 21. *If δ is a limit of Woodin cardinals and there is a measurable cardinal above δ, then for each $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ and $\gamma < \delta$, there exists a γ-weakly-homogeneous tree T such that $p[T] = A$.*

Lemma 22. *Suppose that $T \subset (\omega \times Z)^{<\omega}$ is a γ^+-weakly-homogeneous tree, $\theta > (2^{|T|})^+$ is regular, $X \prec H(\theta)$, $T, \gamma \in X$, $|X| < \gamma$, and $\bar{a} \in p[T]$. Then there exists $Y \prec H(\theta)$ with $X \subseteq Y$, $X \cap \gamma = Y \cap \gamma$, $|X| = |Y|$, and $\bar{a} \in p[T \cap Y]$.*

Proof. Fix $\langle \mu_{a,b} \mid a, b \in \omega^{<\omega} \rangle$ in X witnessing the γ^+-weak-homogeneity of T. Since $\bar{a} \in p[T]$, $\exists \bar{b}$ such that $\langle \mu_{\bar{a}\restriction i, \bar{b}\restriction i} \mid i < \omega \rangle$ is a countably complete tower. Now let $A_i = \bigcap (X \cap \mu_{\bar{a}\restriction i, \bar{b}\restriction i})$. Each A_i is in $\mu_{\bar{a}\restriction i, \bar{b}\restriction i}$; so there is $\bar{c} \in Z^\omega$ such that $\forall i\ \bar{c}\restriction i \in A_i$. Take $Y = X[\{\bar{c}\restriction i \mid i < \omega\}] := \{f(\bar{c}\restriction i) \mid f \in X, \mathrm{dom}(f) = Z^{<\omega}, i < \omega\}$.

Elementarity of Y follows from an argument similar to the proof of Łós's Theorem (see Theorem 1.1.13 of [6]). To see that $Y \cap \gamma = X \cap \gamma$, note that if $\alpha \in Y \cap \gamma$, then $\alpha = f(c \restriction i)$ for some $f \in X$, $\mathrm{dom} f = Z^i$; but $\mu_{a\restriction i, b\restriction i}$ is γ^+-complete, so f is constant on a set in $\mu_{a\restriction i, b\restriction i}$. But then this constant value is in X, and $f(c\restriction i)$ is this value, since $c\restriction i = \bigcap(\mu_{a\restriction i, b\restriction i} \cap X)$. $\square$

Note that Y in the proof above is in some sense a limit ultrapower of the transitive collapse of X.

The following was first proved by Foreman, Magidor and Shelah from a supercompact cardinal, and later improved by Woodin.

Fact 23. *If δ is Woodin, then* $\mathrm{Coll}(\omega_1, <\delta)$ *forces that* NS_{ω_1} *is presaturated, and hence precipitous.*

Recall that an ideal I on ω_1 is *presaturated* if for every sequence of maximal antichains $\{Q_i \mid i < \omega\} \subset \mathcal{P}(\omega_1) \setminus I$, $\forall A \in I^+$, $\exists B \subseteq A$, $B \in I^+$ such that for all $i < \omega$,

$$|\{E \in Q_i \mid E \cap B \in I^+\}| \leq \aleph_1.$$

The following was proved by Kakuda and Magidor independently [3, 7].

Fact 24. *Any c.c.c. forcing preserves that* NS_{ω_1} *is precipitous.*

Recall the hypotheses of main existence lemma: δ is a limit of Woodin cardinals, there exists a measurable cardinal greater than δ, and A is in $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$. To prove the lemma, let κ be the least Woodin cardinal, and γ the least strong inaccessible above κ. Fix γ^+-weakly-homogeneous trees S, T, with $p[S] = A$, $p[T] = \mathbb{R} \setminus A$. Fix a regular $\theta > (2^{|S|})^+, (2^{|T|})^+$. Let X be a countable elementary submodel of $H(\theta)$, with $S, T, \gamma, \kappa \in X$. Repeatedly apply Lemma 22 above to obtain $Y \prec H(\theta)$ such that $X \subseteq Y$, $X \cap \gamma = Y \cap \gamma$, $A = p[S \cap Y]$, $\mathbb{R} \cap A = p[T \cap Y]$. (Then $|Y \cap \mathrm{Ord}| = 2^\omega$.) Now let N be the transitive collapse of Y, and let $\bar{S}, \bar{T}, \bar{\gamma}, \bar{\kappa}$ be the images of S, T, γ, κ therein. Let h be N-generic for $\mathrm{Coll}(\omega_1, <\bar{\kappa})$ followed by a c.c.c. poset of size 2^{ω_1} to make $\mathrm{MA}_{\aleph_1}$ hold. Then $N[h] \models \mathrm{MA}_{\aleph_1}$ + "NS_{ω_1} is precipitous". Then $N[h]$ is iterable, by Lemma 6. Let M be $(V_{\bar{\gamma}})^{N[h]}$, and let $j : (M, \mathrm{NS}^M_{\omega_1}) \to (M^*, \mathrm{NS}^*_{\omega_1})$ be an iteration. By Lemma 4, this induces an iteration of $(N[h], \mathrm{NS}^{N[h]}_{\omega_1})$ with final model (N^*, I^*) (which we'll also call j). Now, N^* is well-founded, and $p[\bar{S}] \subseteq p[j(\bar{S})]$ and

$p[\bar{T}] \subseteq p[j(\bar{T})]$. But by elementarity, $N^* \models p[j(\bar{S})] \cap p[j(\bar{T})] = \emptyset$, and since N^* is well-founded it is correct about this. Then $p[\bar{S}] = p[j(\bar{S})]$ and $p[\bar{T}] = p[j(\bar{T})]$, so $j(A \cap M) = p[j(\bar{S})] \cap M^* = A \cap M^*$.

Remark 1. *Instead of* $\mathrm{Coll}(\omega_1, < \bar{\kappa})$*, we could have taken* h *to be* N*-generic for any poset in* $V_{\bar{\gamma}}^N$ *such that* $N[h] \models$ *"*$\exists$ *precipitous* I *on* ω_1*", and the rest of the proof would have still gone through.*

The main existence lemma gives not only A-iterable preconditions for any $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$, but also A-iterable preconditions containing any given real x, for any $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$, applying the lemma to the set $\{y \oplus x \mid y \in A\}$. Thus we have shown: if there exist infinitely many Woodin cardinals below a measurable, then $\forall x \in \mathbb{R}$, $\forall A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$, there exists some $\mathbb{P}_{\max}$ condition $\langle (M, I), a\rangle$, with $x \in M$, and (M, I) A-iterable.

We didn't quite show $\langle H(\omega_1)^M, \in, A \cap M\rangle \prec \langle H(\omega_1), \in, A\rangle$. We can do this using $A^\sharp$, or by using not just S, T as above but similar trees for all sets projective in A. We omit this for now.

Given a filter $G \subset \mathbb{P}_{\max}$, A_G denotes the set $\bigcup\{e \mid \exists \langle (N, J), e\rangle \in G\}$. We also omit a proof of the following:

Fact 25 ("The combinatorial heart of the $\mathbb{P}_{\max}$ analysis")**.** *Suppose that for each* $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ *there exists an* A*-iterable* $\mathbb{P}_{\max}$ *precondition* (N, I) *such that*

$$\langle H(\omega_1)^M, \in, A \cap M\rangle \prec \langle H(\omega_1), \in, A\rangle,$$

and suppose that $G \subseteq \mathbb{P}_{\max}$ *is an* $L(\mathbb{R})$*-generic filter. Then* $\forall B \in \mathcal{P}(\omega_1)^{L(\mathbb{R})[G]}$*,* $\exists \langle (M, I), a\rangle \in G$ *such that* $B = j(b)$ *for some* $b \in \mathcal{P}(\omega_1)^M$*, where* j *is the unique iteration of* (M, I) *satisfying* $j(a) = A_G$*.*

In other words, all subsets of ω_1 in extensions come from models in the conditions, and $L(\mathbb{R})[G] = L(\mathbb{R})[A_G]$.

Corollary 26. *Suppose that for each* $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ *there exists an* A*-iterable* $\mathbb{P}_{\max}$ *precondition* (N, I) *such that*

$$\langle H(\omega_1)^M, \in, A \cap M\rangle \prec \langle H(\omega_1), \in, A\rangle,$$

and suppose that $G \subseteq \mathbb{P}_{\max}$ *is an* $L(\mathbb{R})$*-generic filter. Then* $\mathrm{NS}_{\omega_1}^{L(\mathbb{R})[G]}$ *is the collection of all sets of the form* $j(e)$*, where for some* $\langle (M, I), a\rangle \in G$*,* $e \in I$*, and* j *is the iteration of* (M, I) *sending* a *to* A_G*.*

Woodin has shown that the hypotheses of Fact 25 are equivalent to the assertion that AD holds in $L(\mathbb{R})$.

5 Π_2 maximality

Proof of Goal 1 So now fix some Π_2 sentence $\varphi = \forall x \exists y \psi(x, y)$ (in the extended language with two new unary predicates), and some $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$. To show that

$$\langle H(\omega_2), \in, A, \mathrm{NS}_{\omega_1}\rangle^{L(\mathbb{R})^{\mathbb{P}_{\max}}} \models \varphi,$$

it is sufficient to show that for each $\langle (M,I),a\rangle \in \mathbb{P}_{\max}$ and each $b \in H(\omega_2)^M$, there exist $\langle (N, \mathrm{NS}^N_{\omega_1}), e\rangle \in \mathbb{P}_{\max}$ and $j : (M,I) \to (M^*, I^*)$ in N such that $j(a) = e$, $I^* = M^* \cap \mathrm{NS}^N_{\omega_1}$, and

$$\langle H(\omega_2)^N, \in, A \cap N, \mathrm{NS}^N_{\omega_1}\rangle \models \exists d\ \psi(j(b), d).$$

The argument is like the one for existence of conditions.

So suppose $\langle (M,I),a\rangle$ is given. Fix P forcing φ. Let δ be the least Woodin cardinal with $p \in V_\delta$; let κ be the least strong inaccessible above δ. Let S,T be κ^+-weakly-homogeneous trees projecting to A, $\mathbb{R} \setminus A$. Let $\theta > (2^{|S|})^+, (2^{|T|})^+$ be regular. Fix $Y \prec H(\theta)$ with $Y \cap \kappa$ countable, $p[S \cap Y] = A$, $p[T \cap Y] = \mathbb{R} \setminus A$ and $\langle (M,I),a\rangle \in Y$.

Let N be the transitive collapse of Y, and let $\bar{P}$,$\bar{S}$,$\bar{\delta}$,$\bar{\kappa}$ be the respective images of P,S,δ,κ under this collapse. Let h_0 be $\bar{P}$-generic for N. Note that since $P \in V_\delta$, $\bar{\delta}$ remains Woodin in $N[h_0]$. The reinterpretation of A is the projection of $\bar{S}$ in the extension. Thus

$$\langle H(\omega_2)^{N[h_0]}, \in, (p[\bar{S}])^{N[h_0]}, \mathrm{NS}^{N[h_0]}_{\omega_1}\rangle \models \varphi.$$

Pick an iteration j of (M,I) in N such that $j(I) = j(M) \cap \mathrm{NS}^{N[h_0]}_{\omega_1}$. Then there exists a $d \in H(\omega_2)^{N[h_0]}$ such that

$$\langle H(\omega_2)^{N[h_0]}, \in, (p[\bar{S}])^{N[h_0]}, \mathrm{NS}^{N[h_0]}_{\omega_1}\rangle \models \psi(j(b), d).$$

Let h_1 be $N[h_0]$-generic for $\mathrm{Coll}(\omega_1, <\bar{\delta})^{N[h_0]}$ followed by some c.c.c. forcing making $\mathrm{MA}_{\aleph_1}$ hold. Now $\langle ((V_{\bar{\kappa}})^{N[h_0][h_1]}, \mathrm{NS}^{N[h_0][h_1]}_{\omega_1}), j(a)\rangle$ is the desired condition. ■

6 Discussion

Question 1. *You've shown that under these conditions, any forceable Π_2 statement must hold in the $\mathbb{P}_{\max}$ extension. Can you give us some cool examples?*

Answer. One example is φ_{AC}: "For every stationary, costationary $A, B \subseteq \omega_1$, there is some $\gamma \in [\omega_1, \omega_2)$, some bijection $\pi : \omega_1 \to \gamma$ such that

$$\{\alpha < \omega_1 \mid \alpha \in A \leftrightarrow (\pi[\alpha]) \in B\}$$

contains a club." This can be used to get an injection $\mathcal{P}(\omega_1) \hookrightarrow \omega_2$, which shows that the Axiom of Choice holds in the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$.

Also, in some cases one can use $\mathbb{P}_{\max}$ to get Π_2 maximality relative to a given Σ_2 statement; that is, for a given Σ_2 statement for $H(\omega_2)$, you can simultaneously get all Π_2 statements forceably consistent with it.

Another useful aspect: often, the combinatorics of forcing to kill off one thing while preserving another are not clear; the combinatorics of doing the same by an iteration may be much clearer. For instance, an analysis of iterations may help answer the question of whether there exists a Dowker space on ω_1.

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Department of Mathematics
Miami University
Oxford, Ohio 45056
United States
larsonpb@muohio.edu