SET THEORY AND OPERATOR ALGEBRAS

ILIJAS FARAH AND ERIC WOFSEY

These notes are based on the six-hour Appalachian Set Theory workshop given by Ilijas Farah on February 9th, 2008 at Carnegie Mellon University. The first half of the workshop (Sections 1-3) consisted of a review of Hilbert space theory and an introduction to C*-algebras, and the second half (Sections 4–6) outlined a few set-theoretic problems relating to C*-algebras. The number and variety of topics covered in the workshop was unfortunately limited by the available time.

Good general references on Hilbert spaces and C*-algebras include [8], [10], [14], [27], and [35]. An introduction to spectral theory is given in [9]. Most of the omitted proofs can be found in most of these references. For a survey of applications of set theory to operator algebras, see [36].

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1. Hilbert spaces and operators

We begin with a review of the basic properties of operators on a Hilbert space. Throughout we let H denote a complex infinite-dimensional separable Hilbert space, and we let (e_n) be an orthonormal basis for H. For $\xi, \eta \in H$, we denote their inner product by $(\xi|\eta)$. We recall that

$$(\eta|\xi) = \overline{(\xi|\eta)}$$

and

$$\|\xi\| = \sqrt{(\xi|\xi)}.$$

The Cauchy–Schwartz inequality says that

$$|(\xi|\eta)| \leq \|\xi\|\|\eta\|.$$

Example 1.1. *The space*

$$\ell^2 = \ell^2(\mathbb{N}) = \left\{\alpha = (\alpha_k)_{k\in\mathbb{N}} : \alpha_k \in \mathbb{C}, \|\alpha\|^2 = \sum |\alpha_k|^2 < \infty\right\}$$

is a Hilbert space under the inner product $(\alpha|\beta) = \sum \alpha_k \overline{\beta_k}$. *If we define* e^n *by* $e^n_k = \delta_{nk}$ *(the Kronecker's* δ*),* (e^n) *is an orthonormal basis for* ℓ^2. *For any* $\alpha \in \ell^2$, $\alpha = \sum \alpha_n e^n$.

Any Hilbert space has an orthonormal basis, and this can be used to prove that all separable infinite-dimensional Hilbert spaces are isomorphic. Moreover, any two infinite-dimensional Hilbert spaces with the same character density (the minimal cardinality of a dense subset) are isomorphic.

Example 1.2. *If* (X, μ) *is a* σ*-finite measure space,*

$$L^2(X,\mu) = \left\{f : X \to \mathbb{C} \text{ measurable} : \|f\|^2 = \int |f|^2 d\mu < \infty\right\} / \{f : f = 0 \text{ a.e.}\}$$

is a Hilbert space under the inner product $(f|g) = \int f\overline{g}d\mu$.

We will let $a, b, \ldots$ denote linear operators $H \to H$. We recall that

$$\|a\| = \sup\{\|a\xi\| : \xi \in H, \|\xi\| = 1\}.$$

If $\|a\| < \infty$, we say a is *bounded.* An operator is bounded iff it is continuous. We denote the algebra of all bounded operators on H by $\mathcal{B}(H)$ (some authors use $L(H)$), and will assume all operators are bounded. We define the *adjoint* a^* of a to be the unique operator satisfying

$$(a\xi|\eta) = (\xi|a^*\eta)$$

for all $\xi, \eta \in H$. Note that since an element of H is determined by its inner products with all other elements of H (e.g., take an orthonormal basis), an operator a is determined by the values of $(a\xi|\eta)$ for all ξ, η.

Lemma 1.3. *For all a, b we have*

(1) $(a^*)^* = a$
(2) $(ab)^* = b^*a^*$
(3) $\|a\| = \|a^*\|$
(4) $\|ab\| \leq \|a\| \cdot \|b\|$
(5) $\|a^*a\| = \|a\|^2$

Proof. These are all easy calculations. For example, for (5), for $\|\xi\| = 1$,

$$\|a\xi\|^2 = (a\xi|a\xi) = (\xi|a^*a\xi) \leq \|a^*a\|,$$

the inequality holding by Cauchy–Schwartz. Taking the sup over all ξ, we obtain $\|a\|^2 \leq \|a^*a\|$. Conversely,

$$\|a^*a\| \leq \|a^*\|\|a\| = \|a\|^2$$

by (3) and (4). □

The first four parts of this say that $\mathcal{B}(H)$ is a Banach *-algebra, and (5) (sometimes called the *C*-equality*) says that $\mathcal{B}(H)$ is a C*-algebra.

1.1. Normal operators and the spectral theorem.

Example 1.4. *Assume (X, μ) is a σ-finite measure space. If $H = L^2(X, \mu)$ and $f\colon X \to \mathbb{C}$ is bounded and measurable, then*

$$H \ni g \mapsto m_f(g) = fg \in H$$

is a bounded linear operator. We have $\|m_f\| = \|f\|_\infty$ and

$$m_f^* = m_{\bar{f}}.$$

Hence $m_f^* m_f = m_f m_f^* = m_{|f|^2}$. We call operators of this form multiplication operators*.*

If $\Phi\colon H_1 \to H_2$ is an isomorphism between Hilbert spaces, then

$$a \mapsto \operatorname{Ad}\Phi(a) = \Phi a \Phi^{-1}$$

is an isomorphism between $\mathcal{B}(H_1)$ and $\mathcal{B}(H_2)$. The operator $\operatorname{Ad}\Phi(a)$ is just a with its domain and range identified with H_2 via Φ.

An operator a is *normal* if $aa^* = a^*a$. These are the operators that have a nice structure theory, which is summarized in the following theorem.

Theorem 1.5 (Spectral Theorem)**.** *If a is a normal operator then there is a finite measure space (X, μ), a measurable function f on X, and a Hilbert space isomorphism $\Phi\colon L^2(X, \mu) \to H$ such that $\operatorname{Ad}\Phi(m_f) = a$.*

Proof. For an elegant proof using Corollary 2.12 see [9, Theorem 2.4.5]. □

That is, every normal operator is a multiplication operator for some identification of H with an L^2 space. Conversely, every multiplication operator is clearly normal. If X is discrete and μ is counting measure, the characteristic functions of the points of X form an orthonormal basis for $L^2(X, \mu)$ and the spectral theorem says that a is diagonalized by this basis. In general, the spectral theorem says that normal operators are "measurably diagonalizable".

Our stating of the Spectral Theorem is rather premature in the formal sense since we are going to introduce some of the key notions used in its proof later on, in §1.2 and §2.3. This was motivated by the insight that the Spectral Theorem provides to theory of C*-algebras.

An operator a is *self-adjoint* if $a = a^*$. Self-adjoint operators are obviously normal. For any $b \in \mathcal{B}(H)$, the "real" and "imaginary" parts of b, $b_0 = (b + b^*)/2$ and $b_1 = (b - b^*)/2i$ are self-adjoint and satisfy $b = b_0 + ib_1$. Thus any operator is a linear combination of self-adjoint operators. It is easy to check that an operator is normal iff its real and imaginary parts commute, so the normal operators are exactly the linear combinations of commuting self-adjoint operators.

Example 1.6. *The real and imaginary parts of a multiplication operator m_f are $m_{\mathrm{Re}\, f}$ and $m_{\mathrm{Im}\, f}$. A multiplication operator m_f is self-adjoint iff f is real (a.e.). By the spectral theorem, all self-adjoint operators are of this form.*

Lemma 1.7 (Polarization)**.** *For any $a \in \mathcal{B}(H)$ and $\xi, \eta \in H$,*

$$(a\xi|\eta) = \frac{1}{4} \sum_{k=0}^{3} i^k (a(\xi + i^k \eta)|\xi + i^k \eta).$$

Proof. An easy calculation. □

Proposition 1.8. *An operator a is self-adjoint iff $(a\xi|\xi)$ is real for all ξ.*

Proof. First, note that

$$((a - a^*)\xi|\xi) = (a\xi|\xi) - (a^*\xi|\xi) = (a\xi|\xi) - (\xi|a\xi) = (a\xi|\xi) - \overline{(a\xi|\xi)}.$$

Thus $(a\xi|\xi)$ is real for all ξ iff $((a - a^*)\xi|\xi) = 0$ for all ξ. But by polarization, the operator $a - a^*$ is entirely determined by the values $((a - a^*)\xi|\xi)$, so this is equivalent to $a - a^* = 0$. □

An operator b such that $(b\xi|\xi) \geq 0$ for all $\xi \in H$ is *positive*, and we write $b \geq 0$. By Proposition 1.8, positive operators are self-adjoint.

Example 1.9. *A multiplication operator m_f is positive iff $f \geq 0$ (a.e.). By the spectral theorem, all positive operators are of the form m_f.*

Exercise 1.10. *For any self-adjoint $a \in \mathcal{B}(H)$ we can write $a = a_0 - a_1$ for some positive operators a_0 and a_1. (Hint: Use the spectral theorem.)*

Proposition 1.11. *b is positive iff $b = a^*a$ for some (non-unique) a. This a may be chosen to be positive.*

Proof. ($\Leftarrow$) $(a^*a\xi|\xi) = (a\xi|a\xi) = \|a\xi\|^2 \geq 0$.

($\Rightarrow$) If b is positive, by the spectral theorem we may assume $b = m_f$ for $f \geq 0$. Let $a = m_{\sqrt{f}}$. □

We say that $p \in \mathcal{B}(H)$ is a *projection* if $p^2 = p^* = p$.

Lemma 1.12. *p is a projection iff it is the orthogonal projection onto a closed subspace of H.*

Proof. Any linear projection p onto a closed subspace of H satisfies $p = p^2$, and orthogonal projections are exactly those that also satisfy $p = p^*$. Conversely, suppose p is a projection. Then p is self-adjoint, so we can write $p = m_f$ for $f : X \to \mathbb{C}$, and we have $f = f^2 = \bar{f}$. Hence $f(x) \in \{0, 1\}$ for (almost) all x. We then set $A = f^{-1}(\{1\})$, and it is easy to see that p is the orthogonal projection onto the closed subspace $L^2(A) \subseteq L^2(X)$. □

If $E \subseteq H$ is a closed subspace, we denote the projection onto E by proj_E.

We denote the identity operator on H by I (some authors use 1). An operator u is *unitary* if $uu^* = u^*u = I$. This is equivalent to u being invertible and satisfying

$$(\xi|\eta) = (u^*u\xi|\eta) = (u\xi|u\eta)$$

for all $\xi, \eta \in H$. That is, an operator is unitary iff it is a Hilbert space automorphism of H. Unitary operators are obviously normal.

Example 1.13. *A multiplication operator m_f is unitary iff $f\bar{f} = |f|^2 = 1$ (a.e.). By the spectral theorem, all unitaries are of this form.*

An operator v is a *partial isometry* if

$$p = vv^* \text{ and } q = v^*v$$

are both projections. Partial isometries are essentially isomorphisms (isometries) between closed subspaces of H: For every partial isometry v there is a closed subspace H_0 of H such that $v \upharpoonright H_0$ is an isometry and $v \upharpoonright H_0^\perp \equiv 0$. However, as the following example shows, partial isometries need not be normal.

Example 1.14. *Let (e_n) be an orthonormal basis of H. We define the* unilateral shift *S by $S(e_n) = e_{n+1}$ for all n. Then $S^*(e_{n+1}) = e_n$ and $S^*(e_0) = 0$. We have $S^*S = I$ but $SS^* = \mathrm{proj}_{\overline{\mathrm{span}}\{e_n\}_{n\geq 1}}$.*

Any complex number z can be written as $z = re^{i\theta}$ for $r \geq 0$ and $|e^{i\theta}| = 1$. Considering $\mathbb{C}$ as the set of operators on a one-dimensional Hilbert space, there is an analogue of this on an arbitrary Hilbert space.

Theorem 1.15 (Polar Decomposition)**.** *Any $a \in \mathcal{B}(H)$ can be written as $a = bv$ where b is positive and v is a partial isometry.*

Proof. See e.g., [27, Theorem 3.2.17 and Remark 3.2.18]. □

However, this has less value as a structure theorem than than one might think, since b and v may not commute. While positive operators and partial isometries are both fairly easy to understand, polar decomposition does not always make arbitrary operators easy to understand. For example, it is easy to show that positive operators and partial isometries always have nontrivial closed invariant subspaces, but it is a famous open problem whether this is true for all operators.

1.2. **The spectrum of an operator.**

Definition 1.16. *The* spectrum *of an operator a is*

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda I \text{ is not invertible}\}.$$

For a finite-dimensional matrix, the spectrum is the set of eigenvalues.

Example 1.17. *A multiplication operator m_f is invertible iff there is some $\epsilon > 0$ such that $|f| > \epsilon$ (a.e.). Thus since $m_f - \lambda I = m_{f-\lambda}$, $\sigma(m_f)$ is the essential range of f (the set of $\lambda \in \mathbb{C}$ such that for every neighborhood U of λ, $f^{-1}(U)$ has positive measure).*

Lemma 1.18. *If $\|a\| < 1$ then $I - a$ is invertible in $C^*(a, I)$.*

Proof. The series $b = \sum_{n=0}^{\infty} a^n$ is convergent and hence in $C^*(a, I)$. By considering partial sums one sees that $(I - a)b = b(I - a) = I$. □

The following Lemma is an immediate consequence of the Spectral Theorem. However, since its assertions (1) is used in the proof of the latter, we provide its proof.

Lemma 1.19. *Let $a \in \mathcal{B}(H)$.*

(1) *$\sigma(a)$ is a compact subset of $\mathbb{C}$.*
(2) *$\sigma(a^*) = \{\overline{\lambda} : \lambda \in \sigma(a)\}$.*
(3) *If a is normal, then a is self-adjoint iff $\sigma(a) \subseteq \mathbb{R}$.*
(4) *If a is normal, then a is positive iff $\sigma(a) \subseteq [0, \infty)$.*

Proof. (1) If $|\lambda| > \|a\|$ then $a - \lambda \cdot I = \lambda(\frac{1}{\lambda} a - I)$ is invertible by Lemma 1.18, and therefore $\sigma(a)$ is bounded.

We shall now show that the set of invertible elements is open. Fix an invertible a. Since the multiplication is continuous, we can find $\epsilon > 0$ such that for every b in the ϵ-ball centered at a there is c such that both $\|I - bc\| < 1$ and $\|I - cb\| < 1$. By Lemma 1.18 there are d_1 and d_2 such that $bcd_1 = d_2cb = I$. Then we have

$$cd_1 = I \cdot cd_1 = d_2cbcd_1 = d_2c \cdot I = d_2c$$

and therefore $cd_1 = d_2c$ is the inverse of b.

Let a be an arbitrary operator. If $\lambda \notin \sigma(a)$ then by the above there is an $\epsilon > 0$ such that every b in the ϵ-ball centered at $a - \lambda \cdot I$ is invertible. In particular, if $|\lambda' - \lambda| < \epsilon$ then $\lambda' \in \sigma(a)$, concluding the proof that $\sigma(a)$ is compact. □

2. C*-algebras

Definition 2.1. *A* concrete C*-algebra *is a norm-closed *-subalgebra of* $\mathcal{B}(H)$. *If* $X \subseteq \mathcal{B}(H)$, *we write* $C^*(X)$ *for the* C*-*algebra generated by* X.

When talking about C*-algebras, we will always assume everything is "*": subalgebras are *-subalgebras (i.e. closed under involution), homomorphisms are *-homomorphisms (i.e. preserve the involution), etc.

Definition 2.2. *An* (abstract) C*-algebra *is a Banach algebra with involution that satisfies* $\|aa^*\| = \|a\|^2$ *for all* a. *That is, it is a Banach space with a product and involution satisfying Lemma 1.3.*

A C*-algebra is *unital* if it has a unit (multiplicative identity). For unital C*-algebras, we can talk about the spectrum of an element.

Lemma 2.3. *Every* C*-*algebra* A *is contained in a unital* C*-*algebra* $\tilde{A} \cong A \oplus \mathbb{C}$.

Proof. On $A \times \mathbb{C}$ define the operations as follows: $(a, \lambda)(b, \xi) = (ab + \lambda b + \xi a, \lambda\xi)$, $(a, \lambda)^* = (a^*, \bar{\lambda})$ and $\|(a, \lambda)\| = \sup_{\|b\| \leq 1} \|ab + \lambda b\|$ and check that this is still a C*-algebra.

A straightforward calculation shows that $(0, 1)$ is the unit of $\tilde{A}$ and that $A \ni a \mapsto (a, 0) \in \tilde{A}$ is an isomorphic embedding. □

We call $\tilde{A}$ the *unitization* of A. By passing to the unitization, we can talk about the spectrum of an element of a nonunital C*-algebra. The unitization retains many of the properties of the algebra A, and many results are proved by first considering the unitization. However, some caution is advised; for example, the unitization is never a simple algebra.

If A and B are unital and $A \subseteq B$ we say A is a *unital subalgebra* of B if the unit of B belongs to A (that is, B has the same unit as A).

Almost all of our definitions (normal, self-adjoint, projections, etc.) make sense in any C*-algebra. More precisely, for an operator a in a C*-algebra A we say that

1. a is *normal* if $aa^* = a^*a$,
2. a is *self-adjoint* (or *hermitian*) if $a = a^*$,
3. a is a *projection* if $a^2 = a^* = a$,
4. a is *positive* (or $a \geq 0$) if $a = b^*b$ for some b,
5. If A is unital then a is *unitary* if $aa^* = a^*a = I$.

Note that a positive element is automatically self-adjoint. For self-adjoint elements a and b write $a \leq b$ if $b - a$ is positive.

The following result says that the spectrum of an element of a C*-algebra doesn't really depend on the C*-algebra itself, as long as we don't change the unit.

Lemma 2.4. *Suppose* A *is a unital subalgebra of* B *and* $a \in A$ *is normal. Then* $\sigma_A(a) = \sigma_B(a)$, *where* $\sigma_A(a)$ *and* $\sigma_B(a)$ *denote the spectra of* a *as an element of* A *and* B, *respectively.*

Proof. See e.g., [27, Corollary 4.3.16] or [9, Corollary 2 on p. 49]. □

2.1. **Some examples of C*-algebras.**

2.1.1. $C_0(X)$. Let X be a locally compact Hausdorff space. Then

$$C_0(X) = \{f : X \to \mathbb{C} : f \text{ is continuous and vanishes at } \infty\}$$

is a C*-algebra with the involution $f^* = \overline{f}$. Here "vanishes at ∞" means that f extends continuously to the one-point compactification of X such that the extension vanishes at ∞. Equivalently, for any $\epsilon > 0$, there is a compact set $K \subseteq X$ such that $|f(x)| < \epsilon$ for $x \notin K$. In particular, if X itself is compact, all continuous functions vanish at ∞, and we write $C_0(X) = C(X)$.

$C_0(X)$ is abelian, so in particular every element is normal. $C_0(X)$ is unital iff X is compact (iff the constant function 1 vanishes at ∞). The unitization of $C_0(X)$ is $C(X^*)$, where X^* is the one-point compactification of X. For $f \in C_0(X)$, we have:

$$\begin{array}{rll} f \text{ is self-adjoint} & \text{iff} & \mathrm{range}(f) \subseteq \mathbb{R}. \\ f \text{ is positive} & \text{iff} & \mathrm{range}(f) \subseteq [0,\infty). \\ f \text{ is a projection} & \text{iff} & f^2(x) = f(x) = \overline{f(x)} \\ & \text{iff} & \mathrm{range}(f) \subseteq \{0,1\} \\ & \text{iff} & f = \chi_U \text{ for a clopen } U \subseteq X. \end{array}$$

For any $f \in C_0(X)$, $\sigma(f) = \mathrm{range}(f)$.

2.1.2. *Full matrix algebras.* M_n, the set of $n \times n$ complex matrices is a unital C*-algebra. In fact, $M_n \cong \mathcal{B}(\ell_2(n))$, where $\ell^2(n)$ is an n-dimensional Hilbert space.

adjoint, unitary:	the usual meaning.
self-adjoint:	hermitian.
positive:	positively definite.
$\sigma(a)$:	the set of eigenvalues.

The spectral theorem on M_n is the spectral theorem of elementary linear algebra: normal matrices are diagonalizable.

2.1.3. *The algebra of compact operators.* It is equal to[1]

$$\begin{aligned} \mathcal{K}(H) =& C^*(\{a \in \mathcal{B}(H) : a[H] \text{ is finite-dimensional}\}) \\ =& \{a \in \mathcal{B}(H) : a[\text{unit ball}] \text{ is precompact}\} \\ =& \{a \in \mathcal{B}(H) : a[\text{unit ball}] \text{ is compact}\}. \end{aligned}$$

(Note that $\mathcal{K}(H)$ is denoted by $C(H)$ in [26] and by $\mathbf{B}_0(H)$ in [27], by analogy with $C_0(X)$.) We write $r_n = \mathrm{proj}_{\overline{\mathrm{span}}\{e_j | j \leq n\}}$ for a fixed basis $\{e_n\}$ of H. Then for $a \in \mathcal{B}(H)$, the following are equivalent:

(1) $a \in \mathcal{K}(H)$,
(2) $\lim_n \|a(I - r_n)\| = 0$,
(3) $\lim_n \|(I - r_n)a\| = 0$.

[1]The second equality is a nontrivial fact specific to the Hilbert space; see [27, Theorem 3.3.3 (iii)]

Note that if a is self-adjoint then

$$\|a(I-r_n)\| = \|(a(I-r_n))^*\| = \|(I-r_n)a\|.$$

It is not hard to see that $\mathcal{K}(H)$ is a (two-sided) ideal of $\mathcal{B}(H)$.

2.2. $L^\infty(X,\mu)$. If (X,μ) is a σ-finite measure space, then the space $L^\infty(X,\mu)$ of all essentially bounded μ-measurable functions on X can be identified with the space of all multiplication operators (see Example 1.4). Then $L^\infty(X,\mu)$ is concrete C*-algebra acting on $L^2(X,\mu)$. It can be shown that $\|m_f\|$ is equal to the essential supremum of f,

$$\|f\|_\infty = \sup\{t \geq 0 : \mu\{x : |f(x)| > t\} > 0\}.$$

2.2.1. *The Calkin algebra.* This is an example of an abstract C*-algebra. The quotient $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ is called the *Calkin algebra.* It is sometimes denoted by Q or $Q(H)$. We write $\pi : \mathcal{B}(H) \to \mathcal{C}(H)$ for the quotient map. The norm on $\mathcal{C}(H)$ is the usual quotient norm for Banach spaces:

$$\|\pi(a)\| = \inf\{\|b\| : \pi(a) = \pi(b)\}$$

The Calkin algebra turns out to be a very "set-theoretic" C*-algebra, analogous to the Boolean algebra $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$.

2.2.2. *Direct limits.*

Definition 2.5. *If Ω is a directed set, A_i, $i \in \Omega$ are C*-algebras and*

$$\varphi_{i,j} : A_i \to A_j \qquad \text{for } i < j$$

is a commuting family of homomorphisms, we define the direct limit *(also called the* inductive limit*) $A = \varinjlim_i A_i$ by taking the algebraic direct limit and completing it. We define a norm on A by saying that if $a \in A_i$,*

$$\|a\|_A = \lim_j \|\varphi_{i,j}(a)\|_{A_j}.$$

This limit makes sense because the $\varphi_{i,j}$ are all contractions by Lemma 2.10.

2.2.3. *UHF (uniformly hyperfinite) algebras.* For each n, define $\Phi_n : M_{2^n} \to M_{2^{n+1}}$ by

$$\Phi_n(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

We then define the CAR (Canonical Anticommutation Relations) algebra (aka the Fermion algebra, aka M_{2^∞} UHF algebra) as the direct limit $M_{2^\infty} = \varinjlim(M_{2^n}, \Phi_n)$. Alternatively, $M_{2^\infty} = \bigotimes_{n\in\mathbb{N}} M_2$, since $M_{2^{n+1}} = M_{2^n} \otimes M_2$ for each n and $\Phi_n(a) = a \otimes 1_{M_2}$.

Note Φ_n maps diagonal matrices to diagonal matrices, so we can talk about the diagonal elements of M_{2^∞}. These turn out to be isomorphic to the algebra $C(K)$, where K is the Cantor set. Thus we can think of M_{2^∞} as a "noncommutative Cantor set."

It is not difficult to see that for m and n in $\mathbb{N}$ there is a unital homomorphism from M_m into M_n if and only if m divides n. If it exists, then this map is unique up to conjugacy. Direct limits of full matrix algebra are called *UHF algebras* and they were classified by Glimm (the unital case) and Dixmier (the general case) in the 1960s. This was the start of the Elliott classification program of separable unital C*-algebras (see [30], [15]).

Exercise 2.6. *Fix $x \in 2^{\mathbb{N}}$ and let $D_x = \{y \in 2^{\mathbb{N}} : (\forall^\infty n) y(n) = x(n)\}$. Enumerate a basis of H as ξ_y, $y \in D_x$. Let s, t range over functions from a finite subset of $\mathbb{N}$ into $\{0,1\}$. For such s define a partial isometry of H as follows. If $y(m) \neq s(m)$ for some $m \in \operatorname{dom}(s)$ then let $u_s(\xi_y) = 0$. Otherwise, if $y \restriction \operatorname{dom}(s) = s$, then let $z \in 2^{\mathbb{N}}$ be such that $z(n) = 1 - y(n)$ for $n \in \operatorname{dom}(s)$ and $z(n) = y(n)$ for $n \notin \operatorname{dom}(s)$ and set $u_s(\xi_y) = \xi_z$.*

1. *Prove that $u_s^* = u_{\bar{s}}$, where $\operatorname{dom}(\bar{s}) = \operatorname{dom}(s)$ and $\bar{s}(n) = 1 - s(n)$ for all $n \in \operatorname{dom}(s)$.*
2. *Prove that $u_s u_s^*$ is the projection to $\overline{\operatorname{span}}\{\xi_y : y \restriction \operatorname{dom}(s) = \bar{s}\}$ and $u_s^* u_s$ is the projection to $\overline{\operatorname{span}}\{\xi_y : y \restriction \operatorname{dom}(s) = s\}$.*
3. *Let A_x be the C*-algebra generated by u_s as defined above. Prove that A_x is isomorphic to M_{2^∞}.*
4. *Show that the intersection of A_x with the atomic masa (see §4.1) diagonalized by ξ_y, $y \in D_x$, consists of all operators of the form $\sum_y \alpha_y \xi_y$ where $y \mapsto \alpha_y$ is a continuous function.*
5. *Show that for x and y in $2^{\mathbb{N}}$ there is a unitary v of H such that $\operatorname{Ad} v$ sends A_x to A_y if and only if $(\forall^\infty n) x(n) = y(n)$. (Hint: cf. Example 3.19.)*

2.3. **Automatic continuity and the Gelfand transform.**

Lemma 2.7. *If a is normal then $\|a^{2^n}\| = \|a\|^{2^n}$ for all $n \in \mathbb{N}$.*

Proof. Using the C*-equality and normality of a we have

$$\|a^2\| = (\|(a^*)^2 a^2\|)^{1/2} = (\|(a^*a)^*(a^*a)\|)^{1/2} = \|a^*a\| = \|a\|^2.$$

Lemma now follows by a straightforward induction. □

Exercise 2.8. *Find $a \in \mathcal{B}(H)$ such that $\|a\| = 1$ and $a^2 = 0$. (Hint: Choose a to be a partial isometry.)*

It can be proved that a C*-algebra is abelian if and only if it contains no nonzero element a such that $a^2 = 0$ (see [10, II.6.4.14]).

The *spectral radius* of an element a of a C*-algebra is defined as

$$r(a) = \max\{|\lambda| : \lambda \in \sigma(a)\}.$$

Lemma 2.9. *Let A be a C*-algebra and $a \in A$ be normal. Then $\|a\| = r(a)$.*

Sketch of a proof. It can be proved (see [9, Theorem 1.7.3], also the first line of the proof of Lemma 1.19) that for an arbitrary a we have

$$\lim_n \|a^n\|^{1/n} = r(a),$$

in particular, the limit on the left hand side exists. By Lemma 2.9, for a normal a this limit is equal to $\|a\|$. □

Lemma 2.10. *Any homomorphism $\Phi : A \to B$ between C*-algebras is a contraction (in particular, it is continuous).*

Proof. By passing to the unitizations, we may assume A and B are unital and Φ is unital as well (i.e., $\Phi(I) = 1$).

Note that for any $a \in A$, $\sigma(\Phi(a)) \subseteq \sigma(a)$ (by the definition of the spectrum). Thus for a normal, using Lemma 2.9,

$$\begin{aligned}\|a\| &= \sup\{|\lambda| : \lambda \in \sigma(a)\}\\ &\geq \sup\{|\lambda| : \lambda \in \sigma(\Phi(a))\}\\ &= \|\Phi(a)\|.\end{aligned}$$

For general a, aa^* is normal so by the C*-equality we have

$$\|a\| = \sqrt{\|aa^*\|} \geq \sqrt{\|\Phi(aa^*)\|} = \|\Phi(a)\|.$$

□

For a unital abelian C*-algebra A consider its *spectrum*

$$X = \{\phi\colon A \to \mathbb{C} : \phi \text{ is a nonzero homomorphism}\}.$$

By Lemma 2.10 each $\phi \in X$ is a contraction. Also $\phi(I) = 1$, and therefore X is a subset of the unit ball of the Banach space dual A^* of A. It is therefore weak*-compact by the Banach–Alaoglu theorem.

Theorem 2.11. *If A is unital and abelian C*-algebra and X is its spectrum, then $A \cong C(X)$.*

Proof. For $a \in A$ the map $f_a\colon X \to \mathbb{C}$ defined by

$$f_a(\phi) = \phi(a)$$

is continuous in the weak*-topology. The transformation

$$A \ni a \to f_a \in C(X)$$

is the *Gelfand transform* of a. An easy computation shows that the Gelfand transform is a *-homomorphism, and therefore by Lemma 2.10 continuous. We need to show it is an isometry.

For $b \in A$ we claim that b is not invertible if and only if $\phi(b) = 0$ for some $\phi \in X$. Only the direct implication requires a proof. Fix a non-invertible b. The $J_b = \{xb : x \in A\}$ is a proper (two-sided) ideal containing b. Let $J \supseteq J_b$ be a maximal proper two sided (not necessarily closed and not necessarily self-adjoint) ideal. Lemma 1.18 implies that $\|I - c\| \geq 1$ for all $c \in J$. Hence the closure of J is still proper, and by maximality J is a closed ideal. Every closed two-sided ideal in a C*-algebra is automatically self-adjoint (see [8, p.11]). Therefore the quotient map ϕ_J from A to A/J is a *-homomorphism. Since A is abelian, by the maximality of J the algebra A/J is a field. For any $a \in A/J$, Lemma 2.9 implies that $\sigma(a)$ is nonempty, and for any $\lambda \in \sigma(a)$, $a - \lambda I = 0$ since A/J is a field. Thus A/J is generated by I and therefore isomorphic to $\mathbb{C}$, so $\phi_J \in X$. Clearly $\phi_J(b) = 0$.

Therefore range$(f_a) = \sigma(a)$ for all a. Lemma 2.9 implies

$$\|a\| = \max\{|\lambda| : \lambda \in \sigma(a)\} = \|f_a\|.$$

Thus $B = \{f_a : a \in A\}$ is isometric to A. Since it separates the points in X, by the Stone–Weierstrass theorem (e.g., [27, Theorem 4.3.4]) B is norm-dense in $C(X)$, and therefore equal to $C(X)$. □

Recall that $\sigma(a)$ is always a compact subset of $\mathbb{C}$ (Lemma 1.19). Theorem 1.5 (Spectral Theorem), is a consequence of the following Corollary and some standard manipulations; see [9, Theorem 2.4.5].

Corollary 2.12. *If $a \in \mathcal{B}(H)$ is normal then $C^*(a, I) \cong C(\sigma(a))$.*

Proof. Let $C^*(a, I) \cong C(X)$ as in Theorem 2.11. For any $\lambda \in \sigma(a)$, $a - \lambda I$ is not invertible so there exists $\phi_\lambda \in X$ such that $\phi_\lambda(a - \lambda I) = 0$, or $\phi_\lambda(a) = \lambda$. Conversely, if there is $\phi \in X$ such that $\phi(a) = \lambda$, then $\phi(a - \lambda I) = 0$ so $\lambda \in \sigma(a)$. Since any nonzero homomorphism to $\mathbb{C}$ is unital, an element $\phi \in X$ is determined entirely by $\phi(a)$. Since X has the weak* topology, $\phi \mapsto \phi(a)$ is thus a continuous bijection from X to $\sigma(a)$, which is a homeomorphism since X is compact. □

Note that the isomorphism above is canonical and maps a to the identity function on $\sigma(a)$. It follows that for any polynomial p, the isomorphism maps $p(a)$ to the function $z \mapsto p(z)$. More generally, for any continuous function $f : \sigma(a) \to \mathbb{C}$, we can then define $f(a) \in C^*(a, I)$ as the preimage of f under the isomorphism. For example, we can define $|a|$ and if a is self-adjoint then it can be written as a difference of two positive operators as

$$a = \frac{|a| + a}{2} - \frac{|a| - a}{2}.$$

If $a \geq 0$, then we can also define $\sqrt{a}$. Here is another application of the "continuous functional calculus" of Corollary 2.12.

Lemma 2.13. *Every $a \in \mathcal{B}(H)$ is a linear combination of unitaries.*

Proof. By decomposing an arbitrary operator into the positive and negative parts of its real and imaginary parts, it suffices to prove that each positive operator a of norm ≤ 1 is a linear combination of two unitaries, $u = a + i\sqrt{I - a^2}$ and $v = a - i\sqrt{I - a^2}$. Clearly $a = \frac{1}{2}(u + v)$. Since $u = v^*$ and $uv = vu = I$, the conclusion follows. □

3. Positivity, states and the GNS construction

The following is a generalization of the spectral theorem to abstract C*-algebras:

Theorem 3.1 (Gelfand–Naimark)**.** *Every commutative C*-algebra is isomorphic to $C_0(X)$ for a unique locally compact Hausdorff space X. The algebra is unital iff X is compact.*

Proof. By Theorem 2.11, the unitization of A is isomorphic to $C(Y)$ for a compact Hausdorff space Y. If $\phi \in X$ is the unique map whose kernel is equal to A, then $A \cong C_0(Y \setminus \{\phi\})$. Uniqueness of X follows from Theorem 3.13 below. □

In fact, the Gelfand–Naimark theorem is functorial: the category of commutative C*-algebras is dual to the category of locally compact Hausdorff spaces. The space X is a natural generalization of the spectrum of a single element of a C*-algebra.

Recall that $a \in A$ is *positive* if $a = b^*b$ for some $b \in A$. It is not difficult to see that for projections p and q we have $p \leq q$ if and only if $pq = p$ if and only if $qp = p$.

Exercise 3.2. *Which of the following are true for projections p and q and positive a and b?*

1. $pqp \leq p$?
2. $a \leq b$ *implies* $ab = ba$?
3. $p \leq q$ *implies* $pap \leq qaq$?
4. $p \leq q$ *implies* $prp \leq qrq$ *for a projection* r?

(Hint: Only one of the above is true.)

Definition 3.3. *Let A be a unital C*-algebra. A continuous linear functional $\varphi : A \to \mathbb{C}$ is* positive *if $\varphi(a) \geq 0$ for all positive $a \in A$. It is a* state *if it is positive and of norm 1. We denote the space of all states on A by $\mathbb{S}(A)$.*

Example 3.4. *If $\xi \in H$ is a unit vector, define a functional ω_ξ on $\mathcal{B}(H)$ by*

$$\omega_\xi(a) = (a\xi|\xi).$$

Then $\omega_\xi(a) \geq 0$ for a positive a and $\omega_\xi(I) = 1$; hence it is a state. We call a state of this form a vector state.

States satisfy a Cauchy–Schwartz inequality:

$$|\varphi(a^*b)|^2 \leq \varphi(a^*a)\varphi(b^*b).$$

Lemma 3.5. *If φ is a state on A and $0 \leq a \leq 1$ is such that $\varphi(a) = 1$, then $\varphi(b) = \varphi(aba)$ for all b.*

Proof. By Cauchy–Schwartz for states (see the paragraph before Theorem 3.7)

$$|\varphi((I-a)b)| \leq \sqrt{\varphi(I-a)\varphi(b^*b)} = 0.$$

Since $b = ab + (I-a)b$, we have $\varphi(b) = \varphi(ab)$, and similarly $\varphi(ab) = \varphi(aba)$. □

Exercise 3.6. *Prove the following.*

1. *If ϕ is a pure state on $M_n(\mathbb{C})$ then there is a rank one projection p such that $\phi(a) = \phi(pap)$ for all a.*
2. *Identify $M_n(\mathbb{C})$ with $\mathcal{B}(\ell_2^n)$. Show that all pure states of $M_n(\mathbb{C})$ are vector states.*

The basic reason we care about states is that they give us representations of abstract C*-algebras as concrete C*-algebras.

Theorem 3.7 (The GNS construction)**.** *Let φ be a state on A. Then there is a Hilbert space H_φ, a representation $\pi_\varphi : A \to \mathcal{B}(H_\varphi)$, and a unit vector $\xi = \xi_\varphi$ in H_φ such that $\varphi = \omega_\xi \circ \pi_\varphi$.*

Proof. We define an "inner product" on A by $(a|b) = \varphi(b^*a)$. We let $J = \{a : (a|a) = 0\}$, so that $(\cdot|\cdot)$ is actually an inner product on the quotient space A/J. We then define H_φ to be the completion of A/J under the induced norm. For any $a \in A$, $\pi_\varphi(a)$ is then the operator that sends $b + J$ to $ab + J$, and ξ_φ is $I + J$. □

3.1. **Irreducible representations and pure states.**

Exercise 3.8. *Assume ψ_1 and ψ_2 are states on A and $0 < t < 1$ and let*

$$\phi = t\psi_1 + (1-t)\psi_2.$$

(1) *Show that ϕ is a state.*
(2) *Show that $H_\phi \cong H_{\psi_1} \oplus H_{\psi_2}$, with $\pi_\phi(a) = \pi_{\psi_1}(a) + \pi_{\psi_2}(a)$, and $\xi_\phi = \sqrt{t}\xi_{\psi_1} + \sqrt{1-t}\xi_{\psi_2}$. In particular, projections to H_{ψ_1} and H_{ψ_2} commute with $\pi(a)$ for all $a \in A$.*

States form a convex subset of A^*. We say that a state is *pure* if it is an extreme point of $\mathbb{S}(A)$. That is, φ is pure iff

$$\varphi = t\psi_0 + (1-t)\psi_1, \qquad 0 \le t \le 1$$

for ψ_0, $\psi_1 \in \mathbb{S}(A)$ implies $\varphi = \psi_0$ or $\varphi = \psi_1$. We denote the set of all pure states on A by $\mathbb{P}(A)$.

While $\mathbb{S}(A)$ is not weak*-compact, the convex hull of $\mathbb{S}(A) \cup \{0\}$ is, and we can use this to show that the Krein–Milman theorem still applies to $\mathbb{S}(A)$. That is, $\mathbb{S}(A)$ is the weak* closure of the convex hull of $\mathbb{P}(A)$. Since by a form of Hahn–Banach lots of states exist, this says that lots of pure states exist.

The space $\mathbb{P}(A)$ is weak*-compact only for a very restrictive class of C*-algebras, including $\mathcal{K}(H)$ and commutative algebras (see Definition 5.6). For example, for UHF algebras the pure states form a dense subset in the compactum of all states ([20, Theorem 2.8]).

Definition 3.9. *A representation $\pi : A \to \mathcal{B}(H)$ of a C*-algebra is* irreducible *(sometimes called* irrep*) if there is no nontrivial subspace $H_0 \subset H$ such that $\pi(a)H_0 \subseteq H_0$ for all $a \in A$. Such a subspace is said to be* invariant *for $\pi[A]$ or* reducing *for π.*

The easy direction of the following is Exercise 3.8.

Theorem 3.10. *A state φ is pure iff π_φ is irreducible. Every irreducible representation is of the form π_φ for some pure state φ.*

Proof. See e.g., [8, Theorem 1.6.6] or [26, (i) $\Leftrightarrow$ (vi) of Theorem 3.13.2]. □

Example 3.11. *If $A = C(X)$, then (by the Riesz representation theorem) states are the same as probability measures on A (writing $\mu(f) = \int f d\mu$).*

Lemma 3.12. *For a state φ of $C(X)$ the following are equivalent:*

(1) *φ is pure,*
(2) *for a unique $x_\varphi \in X$ we have $\varphi(f) = f(x_\varphi)$*
(3) *$\varphi : C(X) \to \mathbb{C}$ is a homomorphism (φ is "multiplicative").*

Proof. Omitted (but see the proof of Theorem 2.11). □

Theorem 3.13. *$\mathbb{P}(C(X)) \cong X$.*

Proof. By (2) in Lemma 3.12, there is a natural map $F : \mathbb{P}(C(X)) \to X$. By (3), it is not hard to show that F is surjective, and it follows from Urysohn's lemma that F is a homeomorphism. □

Proposition 3.14. *For any unit vector $\xi \in \mathcal{B}(H)$, the vector state $\omega_\xi \in \mathbb{S}(\mathcal{B}(H))$ is pure.*

Proof. Immediate from Theorem 3.10. □

Definition 3.15. *We say $\varphi \in \mathbb{S}(\mathcal{B}(H))$ is* singular *if $\varphi[\mathcal{K}(H)] = \{0\}$.*

By factoring through the quotient map $\pi : \mathcal{B}(H) \to \mathcal{C}(H)$, the space of singular states is isomorphic to the space of states on the Calkin algebra $\mathcal{C}(H)$.

Theorem 3.16. *Each state of $\mathcal{B}(H)$ is a weak*-limit of vector states. A pure state is singular iff it is not a vector state.*

Proof. The first sentence is a special case of [19, Lemma 9] when $\mathfrak{A} = \mathcal{B}(H)$. The second sentence is trivial. □

We now take a closer look at the relationship between states and representations of a C*-algebra.

Definition 3.17. *Let A be a C*-algebra and $\pi_i : A \to \mathcal{B}(H_i)$ ($i = 1, 2$) be representations of A. We say π_1 and π_2 are* (unitarily) equivalent *and write $\pi_1 \sim \pi_2$ if there is a unitary (Hilbert space isomorphism) $u\colon H_1 \to H_2$ such that the following commutes:*

$$\begin{tikzcd}
 & \mathcal{B}(H_1) \arrow[dd, "\operatorname{Ad} u"] \\
A \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
 & \mathcal{B}(H_2)
\end{tikzcd}
\qquad \operatorname{Ad} u(a) = uau^*$$

Similarly, if $\varphi_i \in \mathbb{P}(A)$, we say $\varphi_1 \sim \varphi_2$ if there is a unitary $u \in \tilde{A}$ such that the following commutes:

$$\begin{tikzcd} A \arrow[dd, "\operatorname{Ad} u"'] \arrow[dr, "\varphi_1"] & \\ & \mathbb{C} \\ A \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

Proposition 3.18. *For $\varphi_i \in \mathbb{P}(A)$, $\varphi_1 \sim \varphi_2$ iff $\pi_{\varphi_1} \sim \pi_{\varphi_2}$.*

Proof. The direct implication is easy and the converse is a consequence of the remarkable Kadison's Transitivity Theorem. For the proof see e.g., [26, the second sentence of Proposition 3.13.4]. □

3.2. **On the existence of states.** States on an abelian C*-algebra $C(X)$ correspond to probability Borel measures on X (see Example 3.11).

Example 3.19. *On M_2, the following are pure states:*

$$\varphi_0 : \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto a_{11}$$

$$\varphi_1 : \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto a_{22}$$

For any $f \in 2^{\mathbb{N}}$, $\varphi_f = \bigotimes_n \varphi_{f(n)}$ is a pure state on $\bigotimes M_2 = M_{2^\infty}$. Furthermore, one can show that φ_f and φ_g are equivalent iff f and g differ at only finitely many points, and that $\|\varphi_f - \varphi_g\| = 2$ for $f \neq g$. See [26, §6.5] *for a more general setting and proofs.*

Lemma 3.20. *If ϕ is a linear functional of norm 1 on a unital C*-algebra then ϕ is a state if and only if $\phi(I) = 1$.*

Proof. Only the converse implication requires a proof. Assume ϕ is not a state and fix $a \geq 0$. Algebra $C^*(a, I)$ is abelian, and by the Riesz representation theorem the restriction of ϕ to this algebra is given by a Borel measure μ on $\sigma(a)$. The assumption that $\phi(I) = \|\phi\|$ translates as $|\mu| = \mu$, hence μ is a positive probability measure. Since a corresponds to the identity function on $\sigma(a) \subseteq [0, \infty)$ we have $\phi(a) \geq 0$. □

Lemma 3.21. *If A is a subalgebra of B, then any state of B restricts to a state of A, and every (pure) state of A can be extended to a (pure) state of B.*

Proof. The first statement is trivial. Now assume ϕ is a state on $A \subseteq B$. We shall extend ϕ to a state of B under an additional assumption that A is a unital subalgebra of B; the general case is then a straightforward exercise (see Lemma 2.3).

By the Hahn–Banach theorem extend ϕ to a functional ψ on B of norm 1. By Lemma 3.20, ψ is a state of B.

Note that the (nonempty) set of extensions of ϕ to a state of B is weak*-compact and convex. If we start with a pure state φ, then by Krein–Milman the set of extensions of φ to B has an extreme point, which can then be shown to be a pure state on B. □

Lemma 3.22. *For every normal $a \in A$ there is a state ϕ such that $|\phi(a)| = \|a\|$.*

Proof. The algebra $C^*(a)$ is by Corollary 2.12 isomorphic to $C(\sigma(a))$. Consider its state ϕ_0 defined by $\phi_0(f) = f(\lambda)$, where $\lambda \in \sigma(a)$ ia such that $\|a\| = |\lambda|$. This is a pure state and satisfies $|\phi(a)| = \|a\|$.

By Lemma 3.20 extend ϕ_0 to a pure state ϕ on A. □

Exercise 3.23. *Show that there is a C*-algebra A and $a \in A$ such that $|\phi(a)| < \|a\|$ for every state ϕ of A.*

(Hint: First do Exercise 3.6. Then consider $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in $M_2(\mathbb{C})$.)

Theorem 3.24 (Gelfand–Naimark–Segal)**.** *Every C*-algebra A is isomorphic to a concrete C*-algebra.*

Proof. By taking the unitization, we may assume A is unital. Each state φ on A gives a representation π_φ on a Hilbert space H_φ, and we take the product of all these representations to get a single representation $\pi = \bigoplus_{\varphi \in \mathbb{S}(A)} \pi_\varphi$ on $H = \bigoplus H_\varphi$.

We need to check that this representation is faithful, i.e., that $\|\pi(a)\| = \|a\|$ for all a. By Lemma 2.10 we have $\|\pi(a)\| \leq \|a\|$. By Lemma 3.22 for every self-adjoint a we have $|\phi(a)| = \|a\|$.

We claim that $a \neq 0$ implies $\pi(a) \neq 0$. For a we have that $a = b + ic$ for self-adjoint b and c, at least one of which is nonzero. Therefore $\pi(a) = \pi(b) + i\pi(c)$ is nonzero. Thus A is isomorphic to its image $\pi(A) \subseteq \mathcal{B}(H)$, a concrete C*-algebra.

By Lemma 2.10 both π and its inverse are contractions, and therefore π is an isometry. □

Exercise 3.25. *Prove that a separable abstract C*-algebra is isomorphic to a separably acting concrete C*-algebra.*

4. Projections in the Calkin algebra

Recall that $\mathcal{K}(H)$ (see Example 2.1.3) is a (norm-closed two-sided) ideal of $\mathcal{B}(H)$, and the quotient $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ is the *Calkin algebra* (see Example 2.2.1). We write $\pi : \mathcal{B}(H) \to \mathcal{C}(H)$ for the quotient map.

Lemma 4.1. *If $\mathbf{a} \in \mathcal{C}(H)$ is self-adjoint, then there is a self-adjoint $a \in \mathcal{B}(H)$ such that $\mathbf{a} = \pi(a)$.*

Proof. Fix any a_0 such that $\pi(a_0) = \mathbf{a}$. Let $a = (a_0 + a_0^*)/2$. □

Exercise 4.2. *Assume $f: A \to B$ is a *-homomorphism between C*-algebras and p is a projection in the range. Is there necessarily a projection $q \in A$ such that $f(q) = p$? (Hint: Consider the natural *-homomorphism from $C([0,1])$ to $C([0,1/3] \cup [2/3,1])$.)*

Lemma 4.3. *If $\mathbf{p} \in \mathcal{C}(H)$ is a projection, then there is a projection $p \in \mathcal{B}(H)$ such that $\mathbf{p} = \pi(p)$.*

Proof. Fix a self-adjoint a such that $p = \pi(a)$. Represent a as a multiplication operator m_f. Since $\pi(f)$ is a projection, $m_{f^2-f} \in \mathcal{K}(H)$ Let

$$h(x) = \begin{cases} 1, & f(x) \geq 1/2 \\ 0, & f(x) < 1/2. \end{cases}$$

Then m_h is a projection. Also, if (x_α) is such that $f(x_\alpha)^2 - f(x_\alpha) \to 0$, then $h(x_\alpha) - f(x_\alpha) \to 0$. One can show that this implies that since m_{f^2-f} is compact, so is m_{h-f}. Hence $\pi(m_h) = \pi(m_f) = \mathbf{p}$. □

Thus self-adjoints and projections in $\mathcal{C}(H)$ are just self-adjoints and projections in $\mathcal{C}(H)$ modded out by compacts. However, the same is not true for unitaries.

Example 4.4. *Let $S \in \mathcal{B}(H)$ be the unilateral shift (Example 1.14). Then $S^*S = I$ and $SS^* = I - \mathrm{proj}_{\overline{\mathrm{span}}(\{e_0\})} = I - p$. Since p has finite-dimensional range, it is compact, so $\pi(S)^*\pi(S) = I = \pi(S)\pi(S^*)$. That is, $\pi(S)$ is unitary.*

If $\pi(a)$ is invertible, one can define the Fredholm index *of a by*

$$\mathrm{index}(a) = \dim\ker a - \dim\ker a^*.$$

Fredholm index is (whenever defined) invariant under compact perturbations of a ([27, Theorem 3.3.17]*). Since* $\mathrm{index}(u) = 0$ *for any unitary u and* $\mathrm{index}(S) = -1$*, there is no unitary $u \in \mathcal{B}(H)$ such that $\pi(u) = \pi(S)$.*

For A a unital C*-algebra, we write $\mathcal{P}(A)$ for the set of projections in A. We partially order $\mathcal{P}(A)$ by saying $p \leq q$ if $pq = p$. If they exist, we denote joins and meets under this ordering by $p \vee q$ and $p \wedge q$. Note that every $p \in \mathcal{P}(A)$ has a canonical (orthogonal) complement $q = I - p$ such that $p \vee q = I$ and $p \wedge q = 0$.

Lemma 4.5. *Let $p, q \in A$ be projections. Then $pq = p$ iff $qp = p$.*

Proof. Since $p = p^*$ and $q = q^*$, if $pq = p$ then $pq = (pq)^* = q^*p^* = qp$. The converse is similar. □

Lemma 4.6. *Let $p, q \in A$ be projections. Then $pq = qp$ iff pq is a projection, in which case $pq = p \wedge q$ and $p + q - pq = p \vee q$.*

Proof. If $pq = qp$, $(pq)^* = q^*p^* = qp = pq$ and $(pq)^2 = p(qp)q = p^2q^2 = pq$. Conversely, if pq is a projection then $qp = (pq)^* = pq$. Clearly $pq \leq p$ and $pq \leq q$, and if $r \leq p$ and $r \leq q$ then $rpq = (rp)q = rq = r$ so $r \leq pq$.

Hence $pq = p \wedge q$. We similarly have $(1-p)(1-q) = (1-p) \wedge (1-q)$; since $r \mapsto 1-r$ is an order-reversing involution it follows that $p+q-pq = 1-(1-p)(1-q) = p \vee q$. $\square$

For $A = \mathcal{B}(H)$, note that $p \leq q$ iff $\mathrm{range}(p) \subseteq \mathrm{range}(q)$. Also, joins and meets always exist in $\mathcal{B}(H)$ and are given by

$$p \wedge q = \text{ the projection onto } \mathrm{range}(p) \cap \mathrm{range}(q),$$
$$p \vee q = \text{ the projection onto } \overline{\mathrm{span}}(\mathrm{range}(p) \cup \mathrm{range}(q)).$$

That is, $\mathcal{P}(\mathcal{B}(H))$ is a lattice (in fact, it is a complete lattice, as the definitions of joins and meets above generalize naturally to infinite joins and meets).

Note that if X is a connected compact Hausdorff space then $C(X)$ has no projections other than 0 and I.

Proposition 4.7. $\mathcal{B}(H) = C^*(\mathcal{P}(\mathcal{B}(H)))$. *That is, $\mathcal{B}(H)$ is generated by its projections.*

Proof. Since every $a \in \mathcal{B}(H)$ is a linear combination of self-adjoints $a + a^*$ and $i(a - a^*)$, it suffices to show that if b is self-adjoint and $\epsilon > 0$ then there is a linear combination of projections $c = \sum_j \alpha_j p_j$ such that $\|b - c\| < \epsilon$. We may use spectral theorem and approximate m_f by a step function. $\square$

Corollary 4.8. $\mathcal{C}(H) = C^*(\mathcal{P}(\mathcal{C}(H)))$. *That is, $\mathcal{C}(H)$ is generated by its projections.*

Proof. Since a *-homomorphism sends projections to to projections, this is a consequence of Proposition 4.7 $\square$

Proposition 4.9. *Let A be an abelian unital C^*-algebra. Then $\mathcal{P}(A)$ is a Boolean algebra.*

Proof. By Lemma 4.6, commuting projections always have joins and meets, and $p \mapsto I - p$ gives complements. It is then easy to check that this is actually a Boolean algebra using the formulas for joins and meets given by Lemma 4.6. $\square$

By combining Stone duality with Gelfand–Naimark theorem (see the remark after Theorem 3.1) one obtains isomorphism between the categories of Boolean algebras and abelian C*-algebras generated by their projections.

Note that if A is nonabelian, then even if $\mathcal{P}(A)$ is a lattice it may be nondistributive and hence not a Boolean algebra. See also Proposition 4.24 below.

4.1. **Maximal abelian subalgebras.** Since Boolean algebras are easier to deal with than the arbitrary ordering of a poset of projections, we will be interested in abelian (unital) subalgebras of $\mathcal{B}(H)$ and $\mathcal{C}(H)$. In particular, we will look at maximal abelian subalgebras, or "masas." The acronym masa stands for 'Maximal Abelian SubAlgebra' or 'MAximal Self-Adjoint subalgebra.' Pedersen ([27]) uses MAÇA, for 'MAximal Commutative subAlgebra.'

Note that if $H = L^2(X, \mu)$, then $L^\infty(X, \mu)$ is an abelian subalgebra of $\mathcal{B}(H)$ (as multiplication operators).

Theorem 4.10. *$L^\infty(X, \mu) \subset \mathcal{B}(L^2(X, \mu))$ is a masa.*

Proof. See [9, Theorem 4.1.2] or [27, Theorem 4.7.7]. □

Conversely, every masa in $\mathcal{B}(H)$ is of this form. To prove this, we need a stronger form of the spectral theorem, applying to abelian subalgebras rather than just single normal operators.

Theorem 4.11 (General Spectral Theorem)**.** *If A is an abelian subalgebra of $\mathcal{B}(H)$ then there is a finite measure space (X, μ), a subalgebra B of $L^\infty(X, \mu)$, and a Hilbert space isomorphism $\Phi : L^2(X, \mu) \to H$ such that $\operatorname{Ad}\Phi[B] = A$.*

Proof. See [9, Theorem 4.7.13]. □

Corollary 4.12. *For any masa $A \subset \mathcal{B}(H)$, there is a finite measure space (X, μ) and a Hilbert space isomorphism $\Phi : L^2(X, \mu) \to H$ such that*

$$\operatorname{Ad}\Phi[L^\infty(X, \mu)] = A.$$

Proof. By maximality, B must be all of $L^\infty(X)$ in the spectral theorem. □

Example 4.13 (Atomic masa in $\mathcal{B}(H)$)**.** *Fix an orthonormal basis (e_n) for H, which gives an identification $H \cong \ell^2(\mathbb{N}) = \ell^2$. The corresponding masa is then ℓ^∞, or all operators that are diagonalized by the basis (e_n). We call this an* atomic masa *because the corresponding measure space is atomic. The projections in ℓ^∞ are exactly the projections onto subspaces spanned by a subset of $\{e_n\}$. That is, $\mathcal{P}(\ell^\infty) \cong \mathcal{P}(\mathbb{N})$. In particular, if we fix a basis, then the Boolean algebra $\mathcal{P}(\mathbb{N})$ is naturally a sublattice of $\mathcal{P}(\mathcal{B}(H))$. Given $X \subseteq \mathbb{N}$, we write $P_X^{(\vec{e})}$ for the projection onto $\overline{\operatorname{span}}\{e_n : n \in X\}$.*

Example 4.14 (Atomless masa in $\mathcal{B}(H)$)**.** *Let (X, μ) be any atomless finite measure space. Then if we identify H with $L^2(X)$, $L^\infty(X) \subseteq \mathcal{B}(H)$ is a masa, which we call an* atomless masa*. The projections in $L^\infty(X)$ are exactly the characteristic functions of measurable sets, so $\mathcal{P}(L^\infty(X))$ is the measure algebra of (X, μ) (modulo null sets).*

Proposition 4.15. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be an atomless masa. Then $\mathcal{P}(\mathcal{A})$ is isomorphic to the Lebesgue measure algebra of measurable subsets of $[0, 1]$ modulo null sets.*

Proof. Omitted, but see the remark following Proposition 4.9. □

We now relate masas in $\mathcal{B}(H)$ to masas in $\mathcal{C}(H)$.

Theorem 4.16 (Johnson–Parrott, 1972 [23])**.** *If $\mathcal{A}$ is a masa in $\mathcal{B}(H)$ then $\pi[\mathcal{A}]$ is a masa in $\mathcal{C}(H)$.*

Proof. Omitted. □

Theorem 4.17 (Akemann–Weaver [2]). *There exists a masa A in $\mathcal{C}(H)$ that is not of the form $\pi[\mathcal{A}]$ for any masa $\mathcal{A} \subset \mathcal{B}(H)$.*

Proof. By Corollary 4.12, each masa in $\mathcal{B}(H)$ is induced by an isomorphism from H to $L^2(X)$ for a finite measure space X. But the measure algebra of a finite measure space is countably generated, so there are only $2^{\aleph_0}$ isomorphism classes of finite measure spaces. Since H is separable, it follows that there are at most $2^{\aleph_0}$ masas in $\mathcal{B}(H)$.

Now fix an almost disjoint (modulo finite) family $\mathbb{A}$ of infinite subsets of $\mathbb{N}$ of size $2^{\aleph_0}$. Then the projections $p_X = \pi(P_X^{(\vec{e})})$, for $X \in \mathbb{A}$, form a family of orthogonal projections in $\mathcal{C}(H)$. Choose non-commuting projections q_X^0 and q_X^1 in $\mathcal{C}(H)$ below p_X. To each $f\colon \mathbb{A} \to \{0,1\}$ associate a family of orthogonal projections $\{q_X^{f(X)}\}$. Extending each of these families to a masa, we obtain $2^{2^{\aleph_0}}$ distinct masas in $\mathcal{C}(H)$. Therefore some masa in $\mathcal{C}(H)$ is not of the form $\pi[\mathcal{A}]$ for any masa in $\mathcal{B}(H)$. □

Lemma 4.18. *Let $\mathcal{A} \subset \mathcal{B}(H)$ be a masa. Then $J = \mathcal{P}(\mathcal{A}) \cap \mathcal{K}(H)$ is a Boolean ideal in $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\pi[\mathcal{A}]) = \mathcal{P}(\mathcal{A})/J$.*

Proof. It is easy to check that J is an ideal since $\mathcal{K}(H) \subseteq \mathcal{B}(H)$ is an ideal. Let $a \in \mathcal{A}$ be such that $\pi(a)$ is a projection. Writing $\mathcal{A} = L^\infty(X)$, then in the proof of Lemma 4.3, we could have chosen to represent a as a multiplication operator on $L^2(X)$, in which case the projection p that we obtain such that $\pi(p) = \pi(a)$ is also a multiplication operator on $L^2(X)$. That is there is a projection $p \in \mathcal{A}$ such that $\pi(p) = \pi(a)$. Thus $\pi : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\pi[\mathcal{A}])$ is surjective. Furthermore, it is clearly a Boolean homomorphism and its kernel is J, so $\mathcal{P}(\pi[\mathcal{A}]) = \mathcal{P}(\mathcal{A})/J$. □

4.2. **Projections in the Calkin algebra.**

Lemma 4.19. *A projection $p \in \mathcal{B}(H)$ is compact iff its range is finite-dimensional.*

Proof. If we let $B \subseteq H$ be the unit ball, p is compact iff $p[B]$ is precompact. But $p[B]$ is just the unit ball in the range of p, which is (pre)compact iff the range is finite-dimensional. □

Example 4.20. *If $\mathcal{A} = \ell^\infty$ is an atomic masa in $\mathcal{B}(H)$, then we obtain an "atomic" masa $\pi[\mathcal{A}]$ in $\mathcal{C}(H)$. By Lemmas 4.18 and 4.19, $\mathcal{P}(\pi[\mathcal{A}]) \cong \mathcal{P}(\mathbb{N})/\mathrm{Fin}$, where* Fin *is the ideal of finite sets. In particular, if we fix a basis then $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ naturally embeds in $\mathcal{P}(\mathcal{C}(H))$. For this reason, we can think of $\mathcal{P}(\mathcal{C}(H))$ as a "noncommutative" version of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$.*

More generally, one can show that $\mathcal{A} \cap \mathcal{K}(H) = c_0$, the set of sequences converging to 0, so that $\pi[\mathcal{A}] = \ell^\infty/c_0 = C(\beta\mathbb{N} \setminus \mathbb{N})$.

Example 4.21. *If $\mathcal{A}$ is an atomless masa in $\mathcal{B}(H)$, then all of its projections are infinite-dimensional. Thus $\mathcal{P}(\pi[\mathcal{A}]) = \mathcal{P}(\mathcal{A})$. Thus the Lebesgue measure algebra also embeds in $\mathcal{P}(\mathcal{C}(H))$.*

Lemma 4.22. *For projections p and q in $\mathcal{B}(H)$, the following are equivalent:*

(1) $\pi(p) \leq \pi(q)$,
(2) $q(I-p)$ *is compact,*
(3) *For any $\epsilon > 0$, there is a finite-dimensional projection $p_0 \leq I - p$ such that $\|q(I-p-p_0)\| < \epsilon$.*

Proof. The equivalence of (1) and (2) is trivial. For the remaining part see [36, Proposition 3.3]. $\square$

We write $p \leq_{\mathcal{K}} q$ if the conditions of Lemma 4.22 are satisfied. The poset $(\mathcal{P}(\mathcal{C}(H)), \leq)$ is then isomorphic to the quotient $(\mathcal{P}(\mathcal{B}(H)), \leq_{\mathcal{K}})/\sim$, where $p \sim q$ if $p \leq_{\mathcal{K}} q$ and $q \leq_{\mathcal{K}} p$. In the strong operator topology, $\mathcal{P}(\mathcal{B}(H))$ is Polish, and (3) in Lemma 4.22 then shows that $\leq_{\mathcal{K}} \subset \mathcal{P}(\mathcal{B}(H)) \times \mathcal{P}(\mathcal{B}(H))$ is Borel.

Lemma 4.23. *There are projections p and q in $\mathcal{B}(H)$ such that $\pi(p) = \pi(q) \neq 0$ but $p \wedge q = 0$.*

Proof. Fix an orthonormal basis (e_n) for H and let $\alpha_n = 1 - \frac{1}{n}$ and $\beta_n = \sqrt{1-\alpha_n^2}$. Vectors $\xi_n = \alpha_n e_{2n} + \beta_n e_{2n+1}$ for $n \in \mathbb{N}$ are orthonormal and they satisfy $\lim_n (\xi_n | e_{2n}) = 1$. Projections $p = \operatorname{proj}_{\overline{\operatorname{span}}\{e_{2n}:n\in\mathbb{N}\}}$ and $q = \operatorname{proj}_{\overline{\operatorname{span}}\{\xi_n:n\in\mathbb{N}\}}$ are as required. $\square$

Recall that $\mathcal{P}(\mathcal{B}(H))$ is a complete lattice, which is analogous to the fact that $\mathcal{P}(\mathbb{N})$ is a complete Boolean algebra. Since $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ is not a complete Boolean algebra, we would not expect $\mathcal{P}(\mathcal{C}(H))$ to be a complete lattice. More surprisingly, however, the "noncommutativity" of $\mathcal{P}(\mathcal{C}(H))$ makes it not even be a lattice at all.

Proposition 4.24 (Weaver)**.** *$\mathcal{P}(\mathcal{C}(H))$ is not a lattice.*

Proof. Enumerate a basis of H as ξ_{mn}, η_{mn} for m, n in $\mathbb{N}$. Define

$$\zeta_{mn} = \frac{1}{n}\xi_{mn} + \frac{\sqrt{n-1}}{n}\eta_{mn}$$

and

$$\begin{aligned} K =& \overline{\operatorname{span}}\{\xi_{mn} : m, n \in \mathbb{N}\}, & p = \operatorname{proj}_K \\ L =& \overline{\operatorname{span}}\{\zeta_{mn} : m, n \in \mathbb{N}\}, & q = \operatorname{proj}_L. \end{aligned}$$

For $f : \mathbb{N} \to \mathbb{N}$, define

$$M(f) = \overline{\operatorname{span}}\{\xi_{mn} : m \leq f(n)\} \text{ and } r(f) = \operatorname{proj}_{M(f)}.$$

It is easy to show that $r(f) \leq p$ and $r(f) \leq_{\mathcal{K}} q$ for all f, and if $f < g$, then $r(f) <_{\mathcal{K}} r(g)$ strictly. Furthermore, one can show that if $r \leq_{\mathcal{K}} p$ and $r \leq_{\mathcal{K}} q$ then $r \leq_{\mathcal{K}} r(f)$ for some f. In particular, it follows that p and q cannot have a meet under $\leq_{\mathcal{K}}$. $\square$

4.3. **Cardinal invariants.** Since cardinal invariants can often be stated in terms of properties of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ (see [11]), we can look for "noncommutative" (or "quantum") versions of cardinal invariants by looking at analogous properties of $\mathcal{P}(\mathcal{C}(H))$.

Recall that $\mathfrak{a}$ denotes the minimal possible cardinality of a maximal infinite antichain in $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$, or equivalently the minimal possible cardinality of an (infinite) maximal almost disjoint family in $\mathcal{P}(\mathbb{N})$.

Definition 4.25 (Wofsey, 2006 [37])**.** *A family $\mathbb{A} \subseteq \mathcal{P}(\mathcal{B}(H))$ is* almost orthogonal (ao) *if pq is compact for $p \neq q$ in $\mathbb{A}$ but no $p \in \mathbb{A}$ is compact. We define $\mathfrak{a}^*$ to be the minimal possible cardinality of an infinite maximal ao family ("mao family").*

Note that we require every $p \in \mathbb{A}$ to be noncompact since while $\mathrm{Fin} \subset \mathcal{P}(\mathbb{N})$ is only countable, there are $2^{\aleph_0}$ compact projections in $\mathcal{P}(\mathcal{B}(H))$.

Theorem 4.26 (Wofsey, 2006 [37])**.**
1. *It is relatively consistent with ZFC that $\aleph_1 = \mathfrak{a} = \mathfrak{a}^* < 2^{\aleph_0}$,*
2. *MA implies $\mathfrak{a}^* = 2^{\aleph_0}$.*

Proof. Omitted. □

Question 4.27. *Is $\mathfrak{a} = \mathfrak{a}^*$? Is $\mathfrak{a} \geq \mathfrak{a}^*$? Is $\mathfrak{a}^* \geq \mathfrak{a}$?*

It may seem obvious that $\mathfrak{a} \geq \mathfrak{a}^*$, since $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ embeds in $\mathcal{P}(\mathcal{C}(H))$ so any maximal almost disjoint family would give a mao family. However, it turns out that a maximal almost disjoint family can fail to be maximal as an ao family.

An ideal J on $\mathcal{P}(\mathbb{N})$ is a *p-ideal* if for every sequence X_n, $n \in \mathbb{N}$ of elements of J there is $X \in J$ such that $J_n \setminus J$ is finite for all n.

Lemma 4.28 (Steprāns, 2007)**.** *Fix $a \in \mathcal{B}(H)$ and a basis (e_n) for H. Then*

$$J_a = \{X \subseteq \mathbb{N} : P_X^{(\vec{e})} a \text{ is compact}\}$$

is a Borel P-ideal.

Proof. Let $\varphi_a(X) = \|P_X a\|$. This is a lower semicontinuous submeasure on $\mathbb{N}$, and $P_X a$ is compact iff $\lim_n \varphi_a(X \setminus n) = 0$ (see equivalent conditions (1)–(3) in Example 2.1.3). Thus J_a is $F_{\sigma\delta}$. Proving that it is a p-ideal is an easy exercise. □

Proposition 4.29 (Wofsey, 2006 [37])**.** *There is a maximal almost disjoint family $\mathbb{A} \subset \mathcal{P}(\mathbb{N})$ whose image in $\mathcal{P}(\mathcal{B}(H))$ is not a mao family.*

Proof. Let

$$\xi_n = 2^{-n/2} \sum_{j=2^n}^{2^{n+1}-1} e_j.$$

Then ξ_n, $n \in \mathbb{N}$, are orthonormal and $q = \mathrm{proj}_{\overline{\mathrm{span}}\{\xi_n\}}$. Then $\lim_n \|q e_n\| = 0$ hence J_q is a *dense* ideal: every infinite subset of $\mathbb{N}$ has an infinite subset in J_q (choose a sparse enough subset X such that $\sum_{n\in X}\|qe_n\| < \infty$). By density, we can find a maximal almost disjoint family $\mathbb{A}$ that is contained in J_q. Then q is almost orthogonal to P_X for all $X \in \mathbb{A}$, so $\{P_X : X \in \mathbb{A}\}$ is not a mao family. □

In some sense, this is the only way to construct such a counterexample. More precisely, we have the following:

Theorem 4.30. *Let $\mathfrak{a}'$ denote the minimal possible cardinality of a maximal almost disjoint family that is not contained in any proper Borel P-ideal. Then $\mathfrak{a}' \geq \mathfrak{a}$ and $\mathfrak{a}' \geq \mathfrak{a}^*$.*

Proof. The inequality $\mathfrak{a}' \geq \mathfrak{a}$ is trivial, and the inequality $\mathfrak{a}' \geq \mathfrak{a}^*$ follows by Lemma 4.28. □

One can also similarly define other quantum cardinal invariants: $\mathfrak{p}^*, \mathfrak{t}^*, \mathfrak{b}^*$, etc (see e.g., [11]). For example, recall that $\mathfrak{b}$ is the minimal cardinal κ such that there exists a (κ, ω)-gap in $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ and let $\mathfrak{b}$ be the minimal cardinal κ such that there exists a (κ, ω)-gap in $\mathcal{P}(\mathcal{C}(H))$. Considerations similar to those needed in the proof of Proposition 4.24 lead to following.

Theorem 4.31 (Zamora–Avilés, 2007)**.** $\mathfrak{b} = \mathfrak{b}^*$.

Proof. Omitted. □

Almost all other questions about the relationship between these and ordinary cardinal invariants are open. One should also note that equivalent definitions of standard cardinal invariants may lead to distinct quantum cardinal invariants.

4.4. **A Luzin twist of projections.** A question that may be related to cardinal invariants is when collections of commuting projections of $\mathcal{C}(H)$ can be simultaneously lifted to $\mathcal{B}(H)$ such that the lifts still commute. Let $\mathfrak{l}$[2] be the minimal cardinality of such a collection that does not lift (it follows from the proof of Theorem 4.17 that such collections exist). Note that if instead of projections in the definition of $\mathfrak{l}$ we consider arbitrary commuting operators, then its value drops to 2: Consider the unilateral shift and its adjoint (see Example 4.4).

Lemma 4.32. *The cardinal $\mathfrak{l}$ is uncountable. Given any sequence p_i of projections in $\mathcal{B}(H)$ such that $\pi(p_i)$ and $\pi(p_j)$ commute for all i, j, there is an atomic masa $\mathcal{A}$ in $\mathcal{B}(H)$ such that $\pi[\mathcal{A}]$ contains all $\pi(p_i)$.*

Proof. Let $\zeta(i)$, $i \in \mathbb{N}$, be a norm-dense subset of the unit ball of H. We will recursively choose projections q_i in $\mathcal{B}(H)$, orthonormal basis e_i, and $k(j) \in \mathbb{N}$ so that for all $i \leq k(j)$ we have $\pi(q_i) = \pi(p_i)$, $q_i(e_j) \in \{e_j, 0\}$ and $\zeta(j)$ is spanned by $\{e_i : i < k(j)\}$. Assume q_j, $j < n$, and e_i, $i < k(n)$, have been chosen to satisfy these requirements. Let r be the projection to the orthogonal of $\{e_i \mid i < n\}$ and for each $\alpha \in \{1, \perp\}^n$ let $r_\alpha = r_\alpha(n) =$

[2]This symbol is `mathfrak l`

$r\prod_{i<n} q_i^{\alpha(i)}$. For each $\alpha \in \{1, \perp\}^n$ we have that $\pi(p_n)$ and $\pi(r_\alpha)$ commute, hence by Lemma 4.3 there is a projection p_α in $\mathcal{B}(r_\alpha[H])$ such that $\pi(p_\alpha) = \pi(p_n)\pi(r_\alpha)$, and $\pi(p_n) = \sum_\alpha \dot{p}_n \dot{r}_\alpha$. Now pick $k(n+1)$ large enough and unit vectors e_i, $k(n) \leq i < k(n+1)$, each belonging in some $r_\alpha q_n[H]$, such that e_i, $i < k(n+1)$ span $\zeta(n)$.

This assures (e_i) is a basis of H. Let $X(i) = \{n : n \geq k(i)$ and the unique $\alpha \in \{1, \perp\}^n$ such that $e_n \in r_\alpha(n)$ satisfies $\alpha(i) = 1\}$. Fix $i \in \mathbb{N}$. Clearly $q_i = P^{(\vec{e})}_{X(i)}$ satisfies $\pi(q_i) = \pi(p_i)$ and it is diagonalized by (e_n). □

Note that it is not true that any countable collection of commuting projections in $\mathcal{B}(H)$ is simultaneously diagonalizable (e.g., take $H = L^2([0,1])$ and the projections onto $L^2([0,q])$ for each $q \in \mathbb{Q}$).

The following result was inspired by [25].

Proposition 4.33 (Farah, 2006 [17]). *There is a collection of $\aleph_1$ commuting projections in $\mathcal{C}(H)$ that cannot be lifted to simultaneous diagonalizable projections in $\mathcal{B}(H)$.*

Proof. Construct p_ξ $(\xi < \omega_1)$ in $\mathcal{P}(\mathcal{B}(H))$ so that for $\xi \neq \eta$:

(1) $p_\xi p_\eta$ is compact, and
(2) $\|[p_\xi, p_\eta]\| > 1/4$.

Such a family can easily be constructed by recursion using Lemma 4.32.

If there are lifts $P^{(\vec{e})}_{X_\xi}$ of $\pi(p_\xi)$ that are all diagonalized by a basis (e_n), let $d_\xi = p_\xi - P^{(\vec{e})}_{X(\xi)}$. Write $r_n = P^{(\vec{e})}_{\{0,1,\dots,n-1\}}$, so a is compact iff $\lim_n \|a(I - r_n)\| = 0$. By hypothesis, each d_ξ is compact, so fix $\bar{n}$ such that $S = \{\xi : \|d_\xi(I - r_{\bar{n}})\| < 1/8\}$ is uncountable. Since the range of $I - r_{\bar{n}}$ is separable, there are distinct $\xi, \eta \in S$ such that $\|(d_\xi - d_\eta) r_{\bar{n}}\| < 1/8$. But then we can compute that

$$\|[p_\xi, p_\eta]\| \leq \|[P_{X(\xi)}, P_{X(\eta)}]\| + \frac{1}{8} = \frac{1}{4},$$

a contradiction. □

Conjecture 4.34. *The projections constructed in Proposition 4.33 cannot be lifted to simultaneously commuting projections. In particular, $\mathfrak{l} = \aleph_1$.*

4.5. **Maximal chains of projections in the Calkin algebra.** A problem closely related to cardinal invariants is the the description of isomorphism classes of maximal chains in $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ and $\mathcal{P}(\mathcal{C}(H))$. Since $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ is $\aleph_1$-saturated, under CH a back-and-forth argument shows that all maximal chains are order-isomorphic.

Theorem 4.35 (Hadwin, 1998 [21]). *CH implies that any two maximal chains in $\mathcal{P}(\mathcal{C}(H))$ are order-isomorphic.*

Proof. One can show that $\mathcal{P}(\mathcal{C}(H))$ has a similar saturation property and then use the same back-and-forth argument. □

Conjecture 4.36 (Hadwin, 1998 [21])**.** *CH is equivalent to "any two maximal chains in $\mathcal{P}(\mathcal{C}(H))$ are order-isomorphic".*

This conjecture seems unlikely, as the analogous statement for $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ is not true.

Theorem 4.37 (essentially Shelah–Steprāns)**.** *There is a model of $\neg CH$ in which all maximal chains in $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ are isomorphic.*

Proof. Add $\aleph_2$ Cohen reals to a model of CH. We can then build up an isomorphism between any two maximal chains in the generic model in essentially the same way as a nontrivial automorphism of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ is built up in [33]. □

However, this proof cannot be straighforwardly adapted to the case of $\mathcal{P}(\mathcal{C}(H))$.

By forcing towers in $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ of different cofinalities, one can construct maximal chains in $(\mathcal{P}(\mathbb{N}) \setminus \{\mathbb{N}\})/\mathrm{Fin}$ of different cofinalities (in particular, they are non-isomorphic). The same thing works for $\mathcal{P}(\mathcal{C}(H)) \setminus \{\pi(I)\}$.

Theorem 4.38 (Wofsey, 2006 [37])**.** *There is a forcing extension in which there are maximal chains in $\mathcal{P}(\mathcal{C}(H)) \setminus \{\pi(I)\}$ of different cofinalities (and $2^{\aleph_0} = \aleph_2$).*

Proof. Omitted. □

5. Pure states

We now look at some set-theoretic problems concerning pure states on C*-algebras.

Lemma 5.1. *If B is abelian and A is a unital subalgebra of B then any pure state of B restricts to a pure state of A*

Proof. A state on either algebra is pure iff it is multiplicative. It follows that the restriction of a pure state is pure. □

However, in general the restriction of a pure state to a unital subalgebra need not be pure.

Example 5.2. *If ω_ξ is a vector state of $\mathcal{B}(H)$ and $\mathcal{A}$ is the atomic masa diagonalized by a basis (e_n), then $\omega_\xi \restriction \mathcal{A}$ is pure iff $|(\xi|e_n)| = 1$ for some n. Indeed, $\mathcal{A} = \ell^\infty \cong C(\beta\mathbb{N})$, so a state is pure iff it is the limit of the vector states ω_{e_n} under an ultrafilter (see Example 5.24 below).*

Lemma 5.3. *If A is an abelian C*-algebra generated by its projections than a state ϕ of A is pure if and only if $\phi(p) \in \{0,1\}$ for every projection p in A.*

Proof. Let us first consider the case when A is unital. By the Gelfand–Namark theorem we may assume A is $C(X)$ for a compact Hausdorff space X. By Lemma 3.12 a state ϕ of $C(X)$ is pure if and only if there is $x \in X$

such that $\phi(f) = f(x)$ for all f. Such state clearly satisfies $\phi(p) \in \{0, 1\}$ for each projection p in $C(X)$.

If $\phi(p) \in \{0, 1\}$ for every projection p, then $\mathcal{F} = \{p : \phi(p) = 1\}$ is a filter such that for every p either p or $I - p$ is in $\mathcal{F}$. By our assumption, X is zero-dimensional. Therefore $\mathcal{F}$ converges to a point x and Lemma 3.5 implies $\phi(f) = f(x)$ for all $f \in C(X)$.

If A is not unital, then A is isomorphic to $C_0(X)$ for a locally compact Hausdorff space X. Consider it as a subalgebra of $C(\beta X)$ and use an argument similar to the above. □

Proposition 5.4. *Let B be a unital abelian C*-algebra and $A \subseteq B$ be a unital subalgebra. If every pure state of A extends to a unique pure state of B, then $A = B$.*

Proof. We have $B = C(X)$, where X is the space of pure states on B. Since B is abelian, every point of X gives a pure state on A. The hypothesis then says that A separates points of X, so by Stone-Weierstrass $A = C(X)$. □

Without the assumption that B is abelian the conclusion of Proposition 5.4 is no longer true. Let $B = M_{2^\infty}$ and let A be its standard masa—the limit of algebras of diagonal matrices. Then A is isomorphic to $C(2^{\mathbb{N}})$ and each pure state ϕ of A is an evaluation function at some $x \in 2^{\mathbb{N}}$. Assume ψ is a state extension of ϕ to M_{2^∞}. In each M_{2^n} there is a 1-dimensional projection p_n such that $\phi(p_n) = 1$, and therefore Lemma 3.5 implies that for all $a \in M_{2^n}$ we have $\phi(a) = \phi(p_n a p_n) =$ the diagonal entry of the $2^n \times 2^n$ matrix $p_n a p_n$ determined by p_n. Since $\bigcup_n M_{2^n}$ is dense in M_{2^∞}, ψ is uniquely determined by ϕ.

If $A \subseteq B$ are C*-algebras we say that A *separates pure states of* B if for all states $\psi \neq \phi$ of B there is $a \in A$ such that $\phi(a) \neq \psi(a)$.

Problem 5.5 (Noncommutative Stone-Weierstrass problem)**.** *Assume A is a unital subalgebra of B and A separates $\mathbb{P}(B) \cup \{0\}$. Does this necessarily imply $A = B$?*

For more on this problem see e.g., [31].

Definition 5.6 (Kaplansky)**.** *A C*-algebra A is of* type I *if for every irreducible representation $\pi : A \to \mathcal{B}(H)$ we have $\pi[A] \supseteq \mathcal{K}(H)$.*

Type I C*-algebras are also known as GCR, postliminal, postliminary, or smooth. Here GCR stands for 'Generalized CCR' where CCR stands for 'completely continuous representation'; 'completely continuous operators' is an old-fashioned term for compact operators. See [26, §6.2.13] for an amusing explanation of the terminology. Type I C*-algebras should not be confused with type I von Neumann algebras: $\mathcal{B}(H)$ is a type I von Neumann algebra but is not a type I C*-algebra.

Definition 5.7. *A C*-algebra is* simple *if and only if it has no nontrivial (closed two-sided) ideals.*

Recall that the pure states of a C*-algebra correspond to its irreducible representations (Lemma 3.10) and that pure states are equivalent if and only if the corresponding irreducible representations are equivalent (Proposition 3.18).

Lemma 5.8. *A type I C*-algebra has only one pure state up to equivalence if and only if it is isomorphic to $\mathcal{K}(H)$ for some H.*

Proof. The converse direction is a theorem of Naimark, Theorem 5.11 below. For the direct implication, assume A is of type I and all of its pre states are equivalent. It is not difficult to see that A has to be simple. Therefore any irreducible representation is an isomorphism and therefore $\pi[A] = \mathcal{K}(H)$. □

C*-algebras that are not type I are called non-type I or antiliminary. Theorem 5.9 is the key part of Glimm's characterization of type I C*-algebras ([19], see also [26, Theorem 6.8.7]). Its proof contains a germ of what became known as the *Glimm-Effros Dichotomy* ([22]).

Theorem 5.9 (Glimm)**.** *If A is a non-type-I C*-algebra then there is a subalgebra $B \subseteq A$ that has a quotient isomorphic to M_{2^∞}.*

Proof. See [26, §6.8]. □

Corollary 5.10 (Akemann–Weaver, 2002 [1])**.** *If A is non-type-I and has a dense subset of cardinality $< 2^{\aleph_0}$, then A has nonequivalent pure states.*

Proof. By Glimm's Theorem, a quotient of a subalgebra of A is isomorphic to M_{2^∞}, and the pure states φ_f on M_{2^∞} then lift and extend to pure states ψ_f of A. Furthermore, if $f \neq g$ then $\|\psi_f - \psi_g\| = 2$, since the same is true of φ_f and φ_g. In particular, if ψ is any pure state on A, then unitaries that turn ψ into ψ_f must be far apart (distance ≥ 1) from unitaries that turn ψ into ψ_g. Since A does not have a subset of cardinality $2^{\aleph_0}$ such that any two points are far apart from each other, ψ cannot be equivalent to every ψ_f. □

5.1. Naimark's problem.

Theorem 5.11 (Naimark, 1948)**.** *Any two pure states on $\mathcal{K}(H)$ are equivalent, for any (not necessarily separable) Hilbert space H.*

Proof. If ξ and η are unit vectors in H, then the corresponding vector states ω_ξ and ω_η are clearly equivalent, via any unitary that sends ξ to η. Hence the conclusion follows from Proposition 5.12 below. □

Proposition 5.12. *If ϕ is a pure state of $\mathcal{K}(H)$ then there is $\eta \in H$ such that $\phi(a) = \omega_\eta(a) = (a(\eta), \eta)$.*

Proof. An operator $a \in \mathcal{B}(H)$ is a *Trace Class Operator* if for some orthogonal basis E of H we have

$$\sum_{e \in E} (|a|e, e) < \infty.$$

For a trace class operator a define its *trace* as

$$\operatorname{tr}(a) = \sum_{e \in E} (ae, e).$$

Just like in the finite-dimensional case we have $\operatorname{tr}(ab) = \operatorname{tr}(ba)$ for any trace class operator a and any operator b. In particular, this sum does not depend on the choice of the orthonormal basis Every trace class operator is compact since it can be approximated by finite rank operators.

For unit vectors η_1 and η_2 in H define a rank one operator $b_{\eta_1,\eta_2} \colon H \to H$ by

$$b_{\eta_1,\eta_2}(\xi) = (\xi, \eta_2)\eta_1.$$

Claim 5.13. *Given a functional $\phi \in \mathcal{K}(H)^*$ there is a trace class operator u such that $\phi(a) = \operatorname{tr}(ua)$ for all $a \in \mathcal{K}(H)$. If $\phi \geq 0$ then $u \geq 0$.*

Proof. For the existence, see e.g., [27, Theorem 3.4.13]. To see u is positive, pick $\eta \in H$. Then $ub_{\eta,\eta}(\xi) = u((\xi,\eta)\eta) = (\xi,\eta)u(\eta) = b_{u(\eta),\eta}(\xi)$. Therefore

$$\begin{aligned} 0 \leq \phi(b_{\eta,\eta}) = \operatorname{tr}(ub_{\eta,\eta}) = \operatorname{tr}(b_{u(\eta),\eta}) \\ = \sum_{e \in E} (b_{u(\eta),\eta}(e), e) = \sum_{e \in E} (ub_{\eta,\eta}e, e) = (u(\eta), \eta). \end{aligned}$$

(In the last equality we change the basis to E' so that $\eta \in E'$.) □

Since u is a positive compact operator, it is by the Spectral Theorem diagonalizable so we can write $u = \sum_{e \in E} \lambda_e e^*$ with the appropriate choice of the basis E. Thus $\phi(a) = \operatorname{tr}(ua) = \operatorname{tr}(au) = \sum_{e \in E}(aue, e) = \sum_{e \in E} \lambda_e(ae, e) \geq \lambda_{e_0}(ae_0, e_0)$, for any $e_0 \in E$. Since ϕ is a pure state, for each $e \in E$ there is $t_e \in [0,1]$ such that $t_e\phi(a) = \lambda_{e_0}(ae_0, e_0)$. Thus exactly one $t_e = t_{e_0}$ is nonzero, and $a \mapsto \lambda_{e_0}(ae_0, e_0)$. □

Question 5.14 (Naimark, 1951)**.** *If all pure states on a C*-algebra A are equivalent, is A isomorphic to $\mathcal{K}(H)$ for some Hilbert space H?*

Note that by Lemma 5.8 and Corollary 5.10, any counterexample to this must be non-type I and have no dense subset of cardinality $< 2^{\aleph_0}$. A similar argument shows that a counterexample cannot be a subalgebra of $\mathcal{B}(H)$ for a Hilbert space with a dense subset of cardinality $< 2^{\aleph_0}$.

The following lemma is based on recent work of Kishimoto–Ozawa–Sakai and Futamura–Kataoka–Kishimoto.

Lemma 5.15 (Akemann–Weaver, 2004 [1])**.** *Let A be a simple separable unital C*-algebra and let φ and ψ be pure states on A. Then there is a simple separable unital $B \supseteq A$ such that*

(1) *φ and ψ extend to states φ', ψ' on B in a unique way.*
(2) *φ' and ψ' are equivalent.*

Proof. Omitted. □

Recall that $\diamondsuit$ stands for Jensen's diamond principle on ω_1. One of its equivalent reformulations states that there are functions $h_\alpha : \alpha \to \omega_1$, for $\alpha < \omega_1$, such that for every $g : \omega_1 \to \omega_1$, the set $\{\alpha : g \restriction \alpha = h_\alpha\}$ is stationary.

Theorem 5.16 (Akemann–Weaver, 2004 [1])**.** *Assume $\diamondsuit$. Then there is a C*-algebra A, all of whose pure states are equivalent, which is not isomorphic to $\mathcal{K}(H)$ for any H.*

Proof. We construct an increasing chain of simple separable unital C*-algebras A_α ($\alpha \le \omega_1$). We also construct pure states ψ_α on A_α such that for $\alpha < \beta$, $\psi_\beta \restriction A_\alpha = \psi_\alpha$. For each A_α, let $\{\varphi_\alpha^\gamma\}_{\gamma<\omega_1}$ enumerate all of its pure states.

If α is limit, we let $A_\alpha = \varinjlim_{\beta\to\alpha} A_\beta$ and $\psi_\alpha = \varinjlim \psi_\beta$. If α is a successor ordinal, let $A_{\alpha+1} = A_\alpha$. If α is limit and we want to define $A_{\alpha+1}$, suppose there is $\varphi \in \mathbb{P}(A_\alpha)$ such that $\varphi \restriction A_\beta = \varphi_\beta^{h_\alpha(\beta)}$ for all $\beta < \alpha$ (if no such φ exists, let $A_{\alpha+1} = A_\alpha$). Note that $\bigcup_{\beta<\alpha} A_\beta$ is dense in A_α since α is limit, so there is at most one such φ. By Lemma 5.15, let $A_{\alpha+1}$ be such that ψ_α and φ have unique extensions to $A_{\alpha+1}$ that are equivalent, and let $\psi_{\alpha+1}$ be the unique extension of ψ_α.

Let $A = A_{\omega_1}$ and $\psi = \psi_{\omega_1}$. Then A is unital and infinite-dimensional, so A is not isomorphic to any $\mathcal{K}(H)$. Let φ be any pure state on A; we will show that φ is equivalent to ψ, so that A has only one pure state up to equivalence.

Claim 5.17. *$S = \{\alpha : \varphi \restriction A_\alpha$ is pure on $A_\alpha\}$ contains a club.*

Proof. For $x \in A$ and $m \in \mathbb{N}$,

$$T_{m,x} = \left\{\alpha : \exists \psi_1, \psi_2 \in \mathbb{S}(A_\alpha), \varphi \restriction A_\alpha = \frac{\psi_1 + \psi_2}{2} \text{ and } |\varphi(x) - \psi_1(x)| \ge \frac{1}{m}\right\}$$

is bounded in ω_1. Indeed, if it were unbounded, we could take a limit of such ψ_i (with respect to an ultrafilter) to obtain states ψ_i on A such that $\varphi = \frac{\psi_1+\psi_2}{2}$ but such that $|\varphi(x) - \psi_1(x)| \ge \frac{1}{m}$, contradicting purity of φ. Since each A_α is separable, we can take a suitable diagonal intersection of the $T_{m,x}$ over all m and all x in a dense subset of A to obtain a club contained in S. □

Now let $h : S \to \omega_1$ be such that $\varphi \restriction A_\alpha = \varphi_\alpha^{h(\alpha)}$ for all $\alpha \in S$. Since S contains a club, there is some limit ordinal α such that $h \restriction \alpha = h_\alpha$. Then by construction, $\varphi \restriction A_{\alpha+1}$ is equivalent to $\psi_{\alpha+1}$; say $\varphi \restriction A_{\alpha+1} = u\psi_{\alpha+1}u^*$ for a unitary u. For each $\beta \ge \alpha$, ψ_β extends uniquely to $\psi_{\beta+1}$, so by induction we obtain that ψ is the unique extension of $\psi_{\alpha+1}$ to A. Since $\varphi \restriction A_{\alpha+1}$ is equivalent to $\psi_{\alpha+1}$, it also has a unique extension to A, which must be φ. But $u\psi u^*$ is an extension of $\varphi \restriction A_{\alpha+1}$, so $\varphi = u\psi u^*$ and is equivalent to ψ. □

5.2. **Extending pure states on masas.** By Lemma 3.12, a state on an abelian C*-algebra is pure if and only if it is multiplicative, i.e., a *-homomorphism. If the algebra is generated by projections then this is equivalent to asserting that $\phi(p) \in \{0,1\}$ for every projection p (Lemma 5.3).

Definition 5.18. *A masa in a C*-algebra A has the* extension property *(EP) if each of its pure states extends uniquely to a pure state on A.*

On $\mathcal{B}(H)$, every vector state has the unique extension to a pure state. Thus a masa $\mathcal{A} \subset \mathcal{B}(H)$ has the EP iff $\pi[\mathcal{A}] \subset \mathcal{C}(H)$ has the EP (since by Theorem 3.16 all non-vector pure states are singular and thus define pure states on $\mathcal{C}(H)$).

Theorem 5.19 (Kadison–Singer, 1959, [24])**.** *Atomless masas in $\mathcal{B}(H)$ do not have the EP.*

Proof. Omitted. □

Theorem 5.20 (Anderson, 1978 [5])**.** *CH implies there is a masa in $\mathcal{C}(H)$ that has the EP.*

Proof. Omitted. □

Note that Anderson's theorem does not give a masa on $\mathcal{B}(H)$ with the EP, since his masa on $\mathcal{C}(H)$ does not lift to a masa on $\mathcal{B}(H)$. The following is a famous open problem (compare with Problem 5.5).

Problem 5.21 (Kadison–Singer, 1959 [24])**.** *Do atomic masas of $\mathcal{B}(H)$ have the EP?*

This is known to be equivalent to an *arithmetic statement* (i.e., a statement all of whose quantifiers range over natural numbers). As such, it is absolute between transitive models of ZFC and its solution is thus highly unlikely to involve set theory. For more on this problem see [13]. However, there are related questions that seem more set-theoretic. For example, consider the following conjecture:

Conjecture 5.22 (Kadison–Singer, 1959 [24])**.** *For every pure state φ of $\mathcal{B}(H)$ there is a masa $\mathcal{A}$ such that $\varphi \restriction \mathcal{A}$ is multiplicative (i.e., pure).*

We could also make the following stronger conjecture:

Conjecture 5.23. *For every pure state φ of $\mathcal{B}(H)$ there is an* atomic *masa $\mathcal{A}$ such that $\varphi \restriction \mathcal{A}$ is multiplicative.*

Example 5.24. *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and (e_n) be an orthonormal basis for H. Then*

$$\varphi_{\mathcal{U}}^{(\vec{e})}(a) = \lim_{n\to\mathcal{U}} (ae_n|e_n)$$

is a state on $\mathcal{B}(H)$. It is singular iff $\mathcal{U}$ is nonprincipal (if $\{n\} \in \mathcal{U}$, then $\varphi_{\mathcal{U}}^{(\vec{e})} = \omega_{e_n}$).

We say a state of the form $\varphi_{\mathcal{U}}^{(\vec{e})}$ for some basis (e_n) and some ultrafilter $\mathcal{U}$ is diagonalizable. *As noted in Example 5.2, the restriction of a diagonalizable state to the corresponding atomic masa is a pure state of the masa, and every pure state of an atomic masa is of this form.*

Theorem 5.25 (Anderson, 1979 [7])**.** *Diagonalizable states are pure.*

Proof. Omitted. □

Conjecture 5.26 (Anderson, 1981 [4])**.** *Every pure state on $\mathcal{B}(H)$ is diagonalizable.*

Proposition 5.27. *If atomic masas do have the EP, then Anderson's conjecture is equivalent to Conjecture 5.23.*

Proof. If atomic masas have the EP, a pure state on $\mathcal{B}(H)$ is determined by its restriction to any atomic masa on which it is multiplicative. Any multiplicative state on an atomic masa extends to a diagonalizable state, so this means that a pure state restricts to a multiplicative state iff it is diagonalizable. □

We now prove an affirmative answer for a special case of the Kadison-Singer problem. We say an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is a *Q-point* (sometimes called rare ultrafilter) if every partition of $\mathbb{N}$ into finite intervals has a transversal in $\mathcal{U}$. The existence of Q-points is known to be independent from ZFC, but what matters here is that many ultrafilters on $\mathbb{N}$ are not Q-points.

Fix a basis (e_n) and let $\mathcal{A}$ denote the atomic masa of all operators diagonalized by it. In the following proof we write P_X for $P_X^{(\vec{e})}$.

Theorem 5.28 (Reid, 1971 [29])**.** *If $\mathcal{U}$ is a Q-point then the diagonal state $\varphi_{\mathcal{U}} \restriction \mathcal{A}$ has the unique extension to a pure state of $\mathcal{B}(H)$.*

Proof. Fix a pure state φ on $\mathcal{B}(H)$ extending $\varphi_{\mathcal{U}} \restriction \mathcal{A}$ and let $a \in \mathcal{B}(H)$. Without a loss of generality $\mathcal{U}$ is nonprincipal so φ is singular.

Choose finite intervals (J_i) such that $\mathbb{N} = \bigcup_n J_n$ and

$$\|P_{J_m} a P_{J_n}\| < 2^{-m-n}$$

whenever $|m-n| \geq 2$. This is possible by (2) and (3) of Example 2.1.3 since aP_{J_m} and $P_{J_m}a$ are compact. (See [18, Lemma 1.2] for details.) Let $X \in \mathcal{U}$ be such that $X \cap (J_{2i} \cup J_{2i+1}) = \{n(i)\}$ for all i. Then for $Q_i = P_{\{n(i)\}}$ and $f_i = e_{n(i)}$ we have $\varphi(\sum_i Q_i) = 1$ and

$$QaQ = \sum_i Q_i a \sum_i Q_i = \sum_i Q_i a Q_i + \sum_{i \neq j} Q_i a Q_j.$$

The second sum is compact by our choice of (J_i), and $Q_i a Q_i = (af_i|f_i)Q_i$. Now as we make $X \in \mathcal{U}$ smaller and smaller, $\sum_{i \in X}(ae_i|e_i)P_{\{i\}}$ gets closer and closer to $(\lim_{i \to \mathcal{U}}(ae_i|e_i)) \sum P_i = \varphi_{\mathcal{U}}(a) \sum P_i$. Thus

$$\lim_{X \to \mathcal{U}} \pi(P_X a P_X - \varphi_{\mathcal{U}}(a) P_X) \to 0.$$

Since φ is singular and $\varphi(P_X) = \varphi_{\mathcal{U}}(P_X) = 1$, by Lemma 3.5 $\varphi(a) = \varphi(P_X a P_X) = \varphi_{\mathcal{U}}(a)$. Since a was arbitrary, $\varphi = \varphi_{\mathcal{U}}$. □

5.3. **A pure state that is not multiplicative on any masa in $\mathcal{B}(H)$.** The following result shows that Conjecture 5.23 is not true in all models of ZFC.

Theorem 5.29 (Akemann–Weaver, 2005 [2])**.** *CH implies there is a pure state φ on $\mathcal{B}(H)$ that is not multiplicative on any atomic masa.*

Proof. We will sketch the proof of a stronger result (Theorem 5.39) below. □

The basic idea of constructing such a pure state is to encode pure states as "quantum ultrafilters"; a pure state on the atomic masa $\ell^\infty \subset \mathcal{B}(H)$ is equivalent to an ultrafilter. By the following result, states on $\mathcal{B}(H)$ correspond to finitely additive maps from $\mathcal{P}(\mathcal{B}(H))$ into $[0, 1]$.

Theorem 5.30 (Gleason)**.** *Assume $\mu : \mathcal{P}(\mathcal{B}(H)) \to [0, 1]$ is such that $\mu(p+q) = \mu(p) + \mu(q)$ whenever $pq = 0$. Then there is a unique state on $\mathcal{B}(H)$ that extends μ.*

Proof. Omitted. □

We need to go a little further and associate certain 'filters' of projections to pure states of $\mathcal{B}(H)$.

Definition 5.31. *A family $\mathbb{F}$ of projections in a C^*-algebra is a* filter *if*

1. *For any $p, q \in \mathbb{F}$ there is $r \in \mathbb{F}$ such that $r \leq p$ and $r \leq q$.*
2. *If $p \in \mathbb{F}$ and $r \geq p$ then $r \in \mathbb{F}$.*

The filter generated by $\mathbb{X} \subseteq \mathcal{P}(A)$ is the intersection of all filters containing $\mathbb{X}$ (which may not actually be a filter in general if $\mathcal{P}(A)$ is not a lattice).

We say that a filter $\mathcal{F} \subset \mathcal{P}(\mathcal{C}(H))$ *lifts* if there is a commuting family $\mathbb{X} \subseteq \mathcal{P}(\mathcal{B}(H))$ that generates a filter $\mathbb{F}$ such that $\pi[\mathbb{F}] = \mathcal{F}$. Note that, unlike the case of quotient Boolean algebras, $\pi^{-1}[\mathcal{F}]$ itself is not a filter because there exist projections $p, q \in \mathcal{B}(H)$ such that $\pi(p) = \pi(q)$ but $p \wedge q = 0$ (Lemma 4.23).

Question 5.32. *Does every maximal filter $\mathcal{F}$ in $\mathcal{P}(\mathcal{C}(H))$ lift?*

Theorem 5.33 (Anderson, [6])**.** *There are a singular pure state φ of $\mathcal{B}(H)$, an atomic masa $\mathcal{A}_1$, and an atomless masa $\mathcal{A}_2$ such that both $\varphi \restriction \mathcal{A}_1$ and $\varphi \restriction \mathcal{A}_2$ are multiplicative.*

Proof. Omitted. □

Lemma 5.34 (Weaver, 2007)**.** *For $\mathcal{F}$ in $\mathcal{P}(\mathcal{B}(H))$ the following are equivalent:*

(A) *$\|p_1 p_2 \dots p_n\| = 1$ for any $p_1, \dots, p_n \in \mathcal{F}$ and $\mathcal{F}$ is maximal with respect to this property.*

(B) *For all $\epsilon > 0$ and for all finite $F \subseteq \mathcal{F}$ there is a unit vector ξ such that $\|p\xi\| > 1 - \epsilon$ for all $p \in F$.*

Proof. Since $\|p_1 p_2 \dots p_n\| \leq \|p_1\| \cdot \|p_2\| \cdot \dots \cdot \|p_n\| = 1$, clause (A) is equivalent to stating that for every $\epsilon > 0$ there is a unit vector ξ such that $\|p_1 p_2 \dots p_n \xi\| > 1 - \epsilon$. The remaining calculations are left as an exercise to the reader. Keep in mind that, for a projection p, the value of $\|p\xi\|$ is close to $\|\xi\|$ if and only if $\|\xi - p\xi\|$ is close to 0. □

We call an $\mathcal{F}$ satisfying the conditions of Lemma 5.34 a *quantum filter.*

Theorem 5.35 (Farah–Weaver, 2007)**.** *Let $\mathcal{F} \subseteq \mathcal{P}(\mathcal{C}(H))$. Then the following are equivalent:*

(1) *$\mathcal{F}$ is a maximal quantum filter,*
(2) *$\mathcal{F} = \mathcal{F}_\varphi = \{p : \varphi(p) = 1\}$ for some pure state φ.*

Proof. (1⇒2): For a finite $F \subseteq \mathcal{F}$ and $\epsilon > 0$ let

$$X_{F,\epsilon} = \{\varphi \in \mathbb{S}(\mathcal{B}(H)) : \varphi(p) \geq 1 - \epsilon \text{ for all } p \in F\}.$$

If ξ is as in (B) then $\omega_\xi \in X_{F,\epsilon}$.

Since $X_{F,\epsilon}$ is weak*-compact, $\bigcap_{(F,\epsilon)} X_{F,\epsilon} \neq \emptyset$, and any extreme point of the intersection is a pure state with the desired property.[3]

(2⇒1). If $\varphi(p_j) = 1$ for $j = 1, \dots, k$, then $\varphi(p_1 p_2 \dots p_k) = 1$ by Lemma 3.5, hence (A) holds. It is then not hard to show that $\mathcal{F}_\varphi$ also satisfies (B) and is maximal. □

Lemma 5.36. *Let $\mathcal{F}$ be a maximal quantum filter, let (ξ_n) be an orthonormal basis, and let $\mathbb{N} = \bigcup_{j=1}^n A_j$ be a finite partition. Then if there is a $q \in \mathcal{F}$ such that $\|P^{(\vec{\xi})}_{A_j} q\| < 1$ for all j, $\mathcal{F}$ is not diagonalized by (ξ_n) (i.e., the corresponding pure state is not diagonalized by (ξ_n)).*

Proof. Assume $\mathcal{F}$ is diagonalized by (ξ_n) and let $\mathcal{U}$ be such that $\mathcal{F} = \varphi^{(\vec{\xi})}_{\mathcal{U}}$. Then $A_j \in \mathcal{U}$ for some j, but $\|P^{(\vec{\xi})}_{A_j} q\| < 1$ for $q \in \mathcal{F}$, contradicting the assumption that $\mathcal{F}$ is a filter. □

Lemma 5.37. *Let (e_n) and (ξ_n) be orthonormal bases. Then there is a partition of $\mathbb{N}$ into finite intervals (J_n) such that for all k,*

$$\xi_k \in \overline{\operatorname{span}}\{e_i : i \in J_n \cup J_{n+1}\}$$

(modulo a small perturbation of ξ_k) for some $n = n(k)$.

Proof. Omitted. □

For (J_n) as in Lemma 5.37 let

$$D_{\vec{J}} = \{q : \|P^{(\vec{e})}_{J_n \cup J_{n+1}} q\| < 1/2 \text{ for all } n\}$$

[3] It can be proved, using a version of Kadison's Transitivity Theorem, that this intersection is actually a singleton.

Lemma 5.38. *Each $D_{\vec{J}}$ is dense in $\mathcal{P}(\mathcal{C}(H))$, in the sense that for any noncompact $p \in \mathcal{P}(\mathcal{B}(H))$, there is a noncompact $q \leq p$ such that $q \in D_{\vec{J}}$.*

Proof. Taking a basis for range(p), we can thin out the basis and take appropriate linear combinations to find such a q. □

Recall that $\mathfrak{d}$ is the minimal cardinality of a cofinal subset of $\mathbb{N}^{\mathbb{N}}$ under the pointwise order, and we write $\mathfrak{t}^*$ for the minimal length of a maximal decreasing well-ordered chain in $\mathcal{P}(\mathcal{C}(H)) \setminus \{0\}$. In particular, CH (or MA) implies that $\mathfrak{d} = \mathfrak{t}^* = 2^{\aleph_0}$.

Theorem 5.39 (Farah–Weaver)**.** *Assume $\mathfrak{d} \leq \mathfrak{t}^*$.*[4] *Then there exists a pure state on $\mathcal{B}(H)$ that is not diagonalized by any atomic masa.*

Proof. We construct a corresponding maximal quantum filter. By the density of $D_{\vec{J}}$ and $\mathfrak{d} \leq \mathfrak{t}^*$, it is possible to construct a maximal quantum filter $\mathcal{F}$ such that $\mathcal{F} \cap D_{\vec{J}} \neq \emptyset$ for all $\vec{J}$. Given a basis (ξ_k), pick (J_n) such that $\xi_k \in J_{n(k)} \cup J_{n(k)+1}$ (modulo a small perturbation) for all k. Let $A_i = \{k \mid n(k) \mod 4 = i\}$ for $0 \leq i < 4$. Then if $q \in \mathcal{F} \cap D_{\vec{J}}$, $\|P^{(\vec{\xi})}_{A_i} q\| < 1$ for each i. By Lemma 5.36, $\mathcal{F}$ is not diagonalized by (ξ_n). □

6. Automorphisms of the Calkin algebra

We now investigate whether the Calkin algebra has nontrivial automorphisms, which is analogous to the question of whether $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ has nontrivial automorphisms. We say an automorphism Φ of a C*-algebra is *inner* if $\Phi = \operatorname{Ad} u$ for some unitary u.

Example 6.1. *If $A = C_0(X)$ is abelian then each automorphism is of the form*

$$f \mapsto f \circ \Psi$$

for an autohomeomorphism Ψ of X. This automorphism is inner iff Ψ is the identity (because $\operatorname{Ad} u(a) = uau^* = uu^*a = a$ for any u, a). Thus abelian C-algebras often have many outer automorphisms. However, there do exist (locally) compact Hausdorff spaces with no nontrivial autohomeomorphisms (see the introduction of* [28]*), so some nontrivial abelian C*-algebras have no outer automorphisms.*

Proposition 6.2. *All automorphisms of $\mathcal{B}(H)$ are inner.*

Proof. Omitted, but not too different from the proof that each automorphism of $\mathcal{P}(\mathbb{N})$ is given by a permutation of $\mathbb{N}$. □

Proposition 6.3. *The CAR algebra $M_{2^\infty} = \bigotimes_n M_2$ has outer automorphisms.*

[4] The sharpest hypothesis would be $\mathfrak{d} <$"the Novák number of $\mathcal{P}(\mathcal{C}(H))$", if it makes sense.

Proof. Let $\Phi = \bigotimes_n \operatorname{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then Φ is outer since $\bigotimes_n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is not in M_{2^∞}. □

We now turn to automorphisms of the Calkin algebra. Although unitaries in $\mathcal{C}(H)$ need not lift to unitaries in $\mathcal{B}(H)$, they do lift to "almost unitaries" (i.e., isomorphisms between finite codimension subspaces of H). In particular, if we let S denote the unilateral shift, one can show that no inner automorphism of $\mathcal{C}(H)$ sends $\pi(S)$ to $\pi(S^*)$.

Theorem 6.4 (Brown–Douglas–Fillmore, 1977 [12])**.** *Let $a, b \in \mathcal{B}(H)$ be normal. Then there is an automorphism of $\mathcal{C}(H)$ mapping $\pi(a)$ to $\pi(b)$ iff there is an inner automorphism of $\mathcal{C}(H)$ mapping $\pi(a)$ to $\pi(b)$.*

Proof. Omitted □

Question 6.5 (Brown–Douglas–Fillmore, 1977 [12, 1.6(ii)])**.** *Is there an automorphism of $\mathcal{C}(H)$ that maps $\pi(S)$ to $\pi(S^*)$? More generally, are there $a, b \in \mathcal{C}(H)$ such that no inner automorphism maps a to b but an outer automorphism does?*

A more basic question, also asked by Brown, Douglas and Fillmore ([12, 1.6(ii)]), is whether outer automorphisms exist at all. The obvious approach to construct an outer automorphism would be to simply take a nontrivial automorphism of $\mathcal{P}(\mathbb{N})/\mathrm{Fin} \subset \mathcal{P}(\mathcal{C}(H))$ and try to extend it to an automorphism of all of $\mathcal{C}(H)$. Unfortunately, this does not work, by the following consequence of a result of Alperin–Covington–Macpherson [3].

Proposition 6.6. *An automorphism of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ extends to an automorphism of $\mathcal{C}(H)$ if and only if it is trivial.*

Proof. Recall that S_∞ is the group of all permutations of $\mathbb{N}$ and let $\mathrm{FS}(S_\infty)$ be its normal subgroup of all permutations which move only finitely many points. By [3], the outer automorphism group of $S_\infty/\mathrm{FS}(S_\infty)$ is infinite cyclic. The description of outer automorphisms given in [3] easily shows that if an automorphism Φ of the Calkin algebra sends the atomic masa to itself, then the restriction of Φ to the group of all unitaries that send the atomic masa to itself is implemented by a unitary of the Calkin algebra. □

The following fact is a major way in which automorphisms of $\mathcal{C}(H)$ differ from automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$.

Proposition 6.7. *An automorphism Φ of $\mathcal{C}(H)$ is inner iff $\Phi \restriction \mathcal{C}(H_0) : \mathcal{C}(H_0) \to \mathcal{C}(H)$ is $\operatorname{Ad} u$ for a unitary $u : H_0 \to H_1 \subseteq H$ for some (any) infinite-dimensional subspace H_0 of H.*

Proof. Fix u such that $\Phi(b) = ubu^*$ for $b \in \mathcal{C}(H_0)$. Fix $v \in \mathcal{C}(H)$ so that $vv^* = \pi(\mathrm{proj}_{H_0})$ and $v^*v = I$. Then

$$\Phi(a) = \Phi(v^*)\Phi(vav^*)\Phi(v) = \Phi(v^*)uvav^*u^*\Phi(v).$$

For $w = \Phi(v^*)uv$, we then have $\Phi(a) = waw^*$. □

That is, an automorphism is trivial iff it is somewhere trivial. This is not true for automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$. See, for example, the second part of [33].

6.1. **An outer automorphism from the Continuum Hypothesis.** Under CH, we might expect to be able to easily modify the proof that $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ has outer automorphisms to obtain an outer automorphism of $\mathcal{C}(H)$. That is, we would build up an automorphism on separable subalgebras of $\mathcal{C}(H)$, diagonalizing so that it avoids each inner automorphism. However, it turns out that this construction faces serious difficulties at limit stages. Nevertheless, the result still holds.

Theorem 6.8 (Phillips–Weaver, 2006 [28])**.** *CH implies that the Calkin algebra has outer automorphisms.*

Proof. The original proof used the approach above, but required very deep C*-algebra machinery to handle limit stages. We instead sketch a more elementary proof, given in [18]. In addition, by an observation of Stefan Geschke, this proof can be easily modified to require not CH but only $2^{\aleph_0} < 2^{\aleph_1}$ and $\mathfrak{d} = \aleph_1$.

Fix a basis (e_n). For a partition of $\mathbb{N}$ into finite intervals $\mathbb{N} = \bigcup_n J_n$, let $E_n = \overline{\mathrm{span}}\{e_i : i \in J_n\}$ and $\mathcal{D}[\vec{J}]$ be the algebra of all operators that have each E_n as an invariant subspace. We write $\vec{J}^{\mathrm{even}} = (J_{2n} \oplus J_{2n+1})_n$ and $\vec{J}^{\mathrm{odd}} = (J_{2n+1} \oplus J_{2n+2})_n$.

Lemma 6.9. *Suppose u is a unitary and $\alpha_n \in \mathbb{C}$, $|\alpha_n| = 1$ for all n. Then if $v = \sum_n \alpha_n P_{J_n} u$, $\mathrm{Ad}\, u$ and $\mathrm{Ad}\, v$ agree on $\mathcal{D}[\vec{J}]$.*

Proof. Without a loss of generality $u = I$. Note that $a \in \mathcal{D}[\vec{J}]$ iff $a = \sum_n P_{J_n} a P_{J_n}$. Thus for $a \in \mathcal{D}[\vec{J}]$,

$$vav^* = \sum_n \alpha_n P_{J_n} P_{J_n} a \overline{\alpha_n} P_{J_n} = \sum_n P_{J_n} a P_{J_n} = a.$$

□

For partitions $\vec{J}$, $\vec{K}$ of $\mathbb{N}$ into finite intervals we say $\vec{J} \ll \vec{K}$ if for all m there is some n such that $J_m \subseteq K_n \cup K_{n+1}$.

Lemma 6.10. *The ordering $\ll$ is σ-directed and cofinally equivalent to $(\mathbb{N}^{\mathbb{N}}, \leq^*)$.*

Proof. See [18, §3.1]. □

We write $\mathcal{DD}[\vec{J}] = \mathcal{D}[\vec{J}^{\mathrm{even}}] \cup \mathcal{D}[\vec{J}^{\mathrm{odd}}]$.

Definition 6.11. *A family $\mathcal{F}$ of pairs $(\vec{J}, u)$ is a* coherent family of unitaries *if*

(1) *$\mathcal{F}_0 = \{\vec{J} : (\vec{J}, u) \in \mathcal{F}$ for some $u\}$ is $\ll$-cofinal and*
(2) *For $\vec{J} \ll \vec{K}$ in $\mathcal{F}_0$, $\mathrm{Ad}\, u_{\vec{J}}$ and $\mathrm{Ad}\, u_{\vec{K}}$ agree on $\mathcal{DD}[\vec{J}]$.*

The following key lemma is not entirely trivial because not every $a \in \mathcal{B}(H)$ is in $\mathcal{D}[\vec{J}]$ for some $\vec{J}$.

Lemma 6.12. *If $\mathcal{F}$ is a coherent family of unitaries then there is the unique automorphism $\Phi_{\mathcal{F}}$ of $\mathcal{C}(H)$ such that $\Phi_{\mathcal{F}}(\pi(a)) = \pi(uau^*)$ for all $(\vec{J}, u) \in \mathcal{F}$ and all $a \in \mathcal{D}[\vec{J}]$.*

Proof. The main point is that any $a \in \mathcal{B}(H)$ can be decomposed as $a = a_0 + a_1 + c$ so that a_0 and a_1 are in $\mathcal{DD}[\vec{E}]$ for some $\vec{E}$ and c is compact ([18, Lemma 1.2]). See [18, Lemma 1.3] for details. □

A coherent family of unitaries is *trivial* if there is $u_0 \in \mathcal{B}(H)$ such that $\operatorname{Ad} u_0$ and $\operatorname{Ad} u$ agree on $\mathcal{D}[\vec{J}]$ for all $(\vec{J}, u) \in \mathcal{F}$. The automorphism $\Phi_{\mathcal{F}}$ is inner iff $\mathcal{F}$ is trivial.

Now by CH construct $\ll$-increasing and cofinal sequence of partitions J^ξ ($\xi < \omega_1$) and diagonal unitaries $\alpha^\xi \in (\mathcal{U}(1))^{\mathbb{N}} \subset \ell^\infty \subset \mathcal{B}(H)$ such that for $\xi < \eta$ such that for $\xi < \eta$, $\operatorname{Ad} \alpha^\xi$ and $\operatorname{Ad} \alpha^\eta$ agree on $\mathcal{D}[\vec{J}^\xi]$. This can be done with Lemma 6.9 and some work, and it can be done in such a way that there are $\aleph_1$ choices to be made in the construction. We thus obtain $2^{\aleph_1}$ different coherent families of unitaries that give $2^{\aleph_1}$ different automorphisms of $\mathcal{C}(H)$. Since there are only $2^{\aleph_0}$ unitaries in $\mathcal{C}(H)$, some of these automorphisms must be outer. □

Note, however, that this construction still does not answer Question 6.5, since the outer automorphisms constructed are locally given by unitaries. The same is true of the automorphisms constructed in Phillips–Weaver's original proof.

6.2. **Todorcevic's Axiom implies all automorphisms are inner.** Shelah ([32]) constructed a forcing extension in which all automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ are trivial. Toward this end he has developed a sophisticated oracle chain condition forcing. His conclusion was later obtained from the Proper Forcing Axiom (Shelah–Steprāns) and from Todorcevic's Axiom (which is one of the axioms known under the name of *Open Coloring Axiom, OCA*) and Martin's Axiom (Veličković). A number of rigidity results along the similar lines has been obtained since (see [16]).

Todorcevic's Axiom, TA [34]. Assume (V, E) is a graph such that $E = \bigcup_{n=0}^\infty A_n \times B_n$ for some subsets A_n, B_n of V. Then one of the following applies.

(1) (V, E) has an uncountable *clique*: $Y \subseteq X$ such that any two vertices in Y are connected by an edge, or
(2) (V, E) is *countably chromatic*: there is a partition $V = \bigcup_{n=0}^\infty X_n$ so that no edge connects two vertices in the same X_n.

Theorem 6.13 (Farah, 2007 [18]). *Todorcevic's Axiom implies all automorphisms of $\mathcal{C}(H)$ are inner.*

Proof. Fix an automorphism Φ. The proof has two components.

(1) TA implies that the restriction of Φ to $\mathcal{D}[\vec{J}]$ is implemented by a unitary for every $\vec{J}$.
(2) TA implies that every coherent family of unitaries is trivial.

The proof of (1) is a bit more complicated than the proof of (2), and both can be found in [18]. Assertion (2) is false under CH (by the proof of Theorem 6.8). On the other hand, we don't know whether (1) is provable without any additional assumption (cf. Theorem 6.6). □

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Department of Mathematics and Statistics, York University, 4700 Keele Street, North York, Ontario, Canada, M3J 1P3

E-mail address: `ifarah@mathstat.yorku.ca`

URL: `http://www.math.yorku.ca/~ifarah`

Department of Mathematics, Washington University in Saint Louis, Saint Louis, MO 63130