Discrete Mathematics 21-228
Combinatorics Review Problems

The best sources of review material for the midterm are your class notes, homeworks, and exams. Below are a few problems if you wish to practice. Some of these problems may be harder than those on the midterm, some easier.

1. Prove that if five points are placed in an equilateral triangle of side length 1, at least two must be within $\frac{1}{2}$ unit of each other. Show that this is not true of four points.

2. (a) Find a second-order linear recurrence satisfied by
$$a_n = 3 \cdot 2^n - 5^n$$
for $n \geq 0$. Don't forget initial conditions.

   (b) Solve:
$$b_n = 2b_{n-1} + b_{n-2}$$
for $n \geq 2, b_0 = 0, b_1 = 1$.

   (c) Find a general solution (in terms of c_0) for
$$c_n = 2c_{n-1} + n + 1, n \geq 1.$$

3. Give a combinatorial proof of the following identity:
$$k^n = \sum_{i=1}^{n} \binom{k}{i} i! S(n, i)$$
(Recall $S(n, i) =$ the number of ways to partition $[n]$ into i parts.)

4. Let $a_n = n^2$. Determine the generating function for this sequence.

5. Determine the generating function for the sequence $h_0, h_1, h_2, \ldots$ where h_n is the number of solutions to the equation
$$e_1 + e_2 + \cdots + e_k = n$$
where $e_1, e_2, \ldots, e_k$ are non-negative odd integers.

6. Assume that the relation A *knows* B is symmetric (i.e. person A knows person B if and only if person B knows person A). Prove that in a group of n people, there must be two people who know the same number of people in the group

7. Solve the recurrence relation $a_n = 4a_{n-2}$ with initial values $a_0 = 0, a_1 = 1$

8. Let A be the set of all bijections from the set $\{1, 2, 3, 4, 5\}$ to the set $\{a, b, c, d, e\}$. How many elements of A satisfy all of the following conditions:
$$f(1) \notin \{a, b\}, f(3) \neq c, \text{and} f(4) \notin \{d, e\}?$$

9. Prove:

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^k \binom{n}{k} = (-1)^k \binom{n-1}{k}$$

10. Prove:

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

11. Prove:

$$\sum_{i=0}^{n} \binom{n}{i} F_i = F_{2n}$$

where F_i is the i^{th} Fibonacci number.

12. Find a recurrence for the number of strings of length n on $\{0,1\}$ do not contain 011 as a substring.

13. more later?