Discrete Mathematics 21-228
Solving Homogeneous Linear Recurrence Relations

1. Definitions and the main theorem

Let $a_0, a_1, a_2, \dots$ be an infinite sequence.

Def: A **recurrence relation** for this sequence is a function $f : \mathbf{R}^{k+1} \to \mathbf{R}$ such that

$$a_n = f(n, a_{n-1}, a_{n-2}, \dots, a_{n-k})$$

for $n = k, k+1, \dots$.

Def: This sequence satisfies a **homogeneous linear recurrence relation of order k** if there exist constants $c_1, c_2, \dots, c_k$ such that

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

for $n = k, k+1, \dots$.

Def: The **characteristic polynomial** of this homogeneous linear recurrence relation is the polynomial

$$x^k - c_1 x^{k-1} + c_2 x^{k-2} + \cdots - c_{k-1} x - c_k$$

Def: The roots of the characteristic polynomial are called the **characteristic roots** of the homogeneous linear recurrence relation.

Theorem: If $q_1, q_2, \dots, q_k$ are the distinct roots of the characteristic polynomial of a homogeneous linear recurrence relation of order k, then

$$a_n = b_1 q_1^n + b_2 q_2^n + \cdots + b_k q_k^n,$$

where $b_1, \dots, b_k$ are constants, is the general solution of the recurrence.
If q is a root of multiplicity s of the characteristic polynomial, then the sequences

$$q^n, nq^n, n^2 q^n, \dots, n^{s-1} q^n$$

each satisfy the recurrence.

2. Solving it

Given a homogeneous linear recurrence relation of order k, we "guess" that the sequence q^n might be a solution. If so, then q must be a root of the characteristic polynomial. Since this polynomial is of degree k, it will have k roots. If these roots are all distinct, then the general solution of the recurrence is given by the theorem above.

Aside: To prove the theorem, we must show two things: (i) any linear combination of solutions to a homogeneous linear recurrence relation will also be a solution of the recurrence, and (ii) all solutions will be of this form. To justify (i) is simple algebra, to justify (ii), note that a recurrence relation of order k is determined by its k initial conditions. The general solution has k variables in it $(b_1, ..., b_k)$, giving us a set of k linear equations with k unknowns, and so there is always a solution to the recurrence that takes this general form.

Note that in the general solution, the q's are responsible for satisfying the recurrence, and the b's are responsible for satisfying the initial conditions.

If the characteristic roots are not distinct, then the general solution takes a more complicated form, but the ideas are the same.

Thus to solve a homogeneous linear recurrence relation of order k, we must:

1. Write down the characteristic polynomial
2. Find the characteristic roots
3. Write down the general solution
4. Use the initial conditions to solve for the unknown constants in the general solution

So the procedure is simple, and for small values of k, it is easy.