Discrete Mathematics 21-228
Midterm Exam Solutions

1. An integer is called *square-free* if it is not divisible by any perfect square other than 1. How many positive integers between 1 and 200 are sqaure-free? (No credit will be given for simply listing these).
Note that if an integer is not square-free, then it is divisible by a prime square. Thus we consider the sets:

$$\begin{aligned} A &= \{x \in \mathbf{Z} : 1 \leq x \leq 200\} \\ P &= \{i \in [14] : i \text{ prime}\} \\ A_i &= \{x \in A : x \text{ is divisible by } i^2, i \in P\} \end{aligned}$$

Then

$$\begin{aligned} |\bigcap_{i\in P} \overline{A_i}| &= |A| - \sum_{i\in P} |A_i| + \sum_{i<j\in P} |A_i \cap A_j| - \sum_{i<j<k\in P} |A_i \cap A_j \cap A_k| + \cdots \\ &= 200 - \sum_{i\in P} \lfloor \frac{200}{i^2} \rfloor + \sum_{i<j\in P} \lfloor \frac{200}{i^2 j^2} \rfloor - \sum_{i<j<k\in P} \lfloor \frac{200}{i^2 j^2 k^2} \rfloor + \cdots \\ &= 200 - (\lfloor \frac{200}{4} \rfloor + \lfloor \frac{200}{9} \rfloor + \cdots + \lfloor \frac{200}{169} \rfloor) + (\lfloor \frac{200}{36} \rfloor + \lfloor \frac{200}{100} \rfloor + \lfloor \frac{200}{198} \rfloor) - 0 + \cdots \\ &= 200 - (50 + 22 + 8 + 4 + 1 + 1) + (5 + 2 + 1) \\ &= 200 - 86 + 8 \\ &= 122 \end{aligned}$$

2. Let a_n denote the number of ways to tile a 3 x $(2n)$ grid of squares using 2 x 1 tiles.

   (a) Derive a recurrence relation for a_n.
   Examining cases, we derive the recurrence:

   $$a_n = 3a_{n-1} + 2a_{n-2} + 2a_{n-3} + \cdots + 2a_1 + 2a_0$$

   To get a more compact recurrence, note that

   $$a_n - a_{n-1} = 3a_{n-1} - a_{n-2}$$

   $$\Rightarrow a_n = 4a_{n-1} - a_{n-2}$$

   for $n \geq 2$, with initial conditions $a_0 = 1, a_1 = 3$

   (b) Solve for a_n explicitly.
   The characteristic polynomial of this recurrence is

   $$q^2 - 4q + 1$$

   and the characteristic roots are $q_1 = 2 + \sqrt{3}$ and $q_2 = 2 - \sqrt{3}$. Thus our general solution is of the form:

   $$a_n = b_1(2 + \sqrt{3})^n + b_2(2 - \sqrt{3})^n$$

Plugging in $n = 0$ and $n = 1$ we get the system

$$\begin{aligned}
1 &= b_1 + b_2 \\
3 &= (2+\sqrt{3})b_1 + (2-\sqrt{3})b_2
\end{aligned}$$

$$\Rightarrow b_1 = \frac{3+\sqrt{3}}{6}, b_2 = \frac{3-\sqrt{3}}{6}$$

So the solution is

$$a_n = \frac{3+\sqrt{3}}{6}(2+\sqrt{3})^n + \frac{3-\sqrt{3}}{6}(2-\sqrt{3})^n$$

(c) Use your recurrence relation to find the generating function for a_n.
Define the generating function for a_n to be

$$A(x) = \sum_{n\geq 0} a_n x^n$$

Multiplying by x^n on both sides of the recurrence, and summing over $n \geq 2$, we get:

$$\sum_{n\geq 2} a_n x^n = \sum_{n\geq 2} 4a_{n-1}x^n - \sum_{n\geq 2} a_{n-2}x^n$$

On the RHS,

$$\sum_{n\geq 2} a_n x^n = A(x) - 3x - 1$$

On the LHS,

$$\begin{aligned}
\sum_{n\geq 2} 4a_{n-1}x^n - \sum_{n\geq 2} a_{n-2}x^n &= 4x\sum_{n\geq 1} a_n x^n - x^2\sum_{n\geq 0} a_n x^n \\
&= 4x(A(x) - 1) - x^2 A(x)
\end{aligned}$$

Thus

$$\begin{aligned}
A(x) - 3x - 1 &= 4x(A(x) - 1) - x^2 A(x) \\
\Rightarrow A(x) &= \frac{1-x}{1-4x+x^2}
\end{aligned}$$

2 (alternate) Let a_n denote the number of ways to tile a 2 x n grid of squares using a combination of 2 x 1 and/or 2 x 2 tiles.

(a) Derive a recurrence relation for a_n.
Examining cases, we derive:

$$a_n = a_{n-1} + 2a_{n-2}$$

for $n \geq 2, a_0 = 1, a_1 = 1$.

(b) Solve for a_n explicitly.
The characteristic polynomial of this recurrence is

$$q^2 - q - 2 = (q-2)(q+1)$$

and the characteristic roots are $q_1 = 2, q_2 = -1$. Thus the general solution is of the form

$$a_n = b_1 2^n + b_2(-1)^n$$

Plugging in $n = 0$ and $n = 1$ we get the system

$$\begin{aligned} 1 &= b_1 + b_2 \\ 1 &= 2b_1 - b_2 \end{aligned}$$

$$\Rightarrow b_1 = \frac{2}{3}, b_2 = \frac{1}{3}$$

So the solution is

$$a_n = \frac{2}{3}2^n + \frac{1}{3}(-1)^n$$

(c) Use your recurrence relation to find the generating function for a_n.
Define the generating function for a_n to be

$$A(x) = \sum_{n \geq 0} a_n x^n$$

Multiplying by x^n on both sides of the recurrence, and summing over $n \geq 2$, we get:

$$\sum_{n\geq 2} a_n x^n = \sum_{n \geq 2} a_{n-1} x^n + \sum_{n \geq 2} 2a_{n-2} x^n$$

On the RHS,

$$\sum_{n \geq 2} a_n x^n = A(x) - x - 1$$

On the LHS,

$$\begin{aligned} \sum_{n \geq 2} a_{n-1} x^n + \sum_{n \geq 2} 2a_{n-2} x^n &= x \sum_{n \geq 1} a_n x^n + 2x^2 \sum_{n \geq 0} a_n x^n \\ &= x(A(x) - 1) + 2x^2 A(x) \end{aligned}$$

Thus

$$\begin{aligned} A(x) - x - 1 &= x(A(x) - 1) + 2x^2 A(x) \\ \Rightarrow A(x) &= \frac{1}{1 - x - 2x^2} \end{aligned}$$

3. Let $f(n, k) =$ the number of permutations of $[n]$ with exactly k cycles.

(a) Derive a formula for each of the following quantities:

i. $f(n,1)$
A permutation with one cycle is just a circular permutation, so $f(n,1) = (n-1)!$

ii. $f(n,n)$
A permutation with n cycles must map each element to itself, so $f(n,n) = 1$

iii. $f(n,n-1)$
A permutation with $n-1$ cycles is uniquely determined by its lone 2-cycle. There are $\binom{n}{2}$ ways to choose 2 elements, and so $f(n,n-1) = \binom{n}{2}$.

iv. $f(n,2)$

$$\begin{aligned} f(n,2) &= \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}(k-1)!(n-k-1)! \\ &= \frac{1}{2}\sum_{k=1}^{n-1}\frac{n!}{k(n-k)} \\ &= \frac{n!}{2}\sum_{k=1}^{n-1}\frac{1}{k(n-k)} \end{aligned}$$

since there are $\binom{n}{k}$ ways to choose k elements, and $(k-1)!(n-k-1)!$ ways to permute a cycle of k elements and a cycle of $n-k$ elements. Counting in this manner we would count every permutation twice (once when choosing the k elements in one cycle, once when choosing the $(n-k)$ elements in the other cycle). Thus we must divide by 2.

(b) Derive a recurrence relation for $f(n,k)$ and use this relation to display all values of $f(n,k)$ in a triangle for $n \leq 6$.
In a permutation of $[n]$, n can be in its own cycle. In this case the other $n-1$ elements are divided among $k-1$ cycles. There are $f(n-1,k-1)$ ways to do this. Alternatively, n could be mapped to another element m. In this case, if we "remove" n from the cycle, we are left with a permutation of $n-1$ elements having k cycles. There are $f(n-1,k)$ ways to do this, and $n-1$ ways to choose m, so there are $(n-1)f(n-1,k)$ permutations of this type. We conclude:

$$f(n,k) = f(n-1,k-1) + (n-1)f(n-1,k)$$

for $n \geq 1, k \geq 1$ with initial conditions as follows:

$$f(0,0) = 1, \quad f(n,0) = 0 \text{ if } n > 0, \quad f(0,k) = 0 \text{ if } k > 0$$

For a table of the values of $f(n,k)$ see *East Side, West Side* page 25.

(c) Prove that for all $n \geq 2, f(n,2) \geq f(n,1)$.
Proof 1: If $n = 2$, then $f(2,2) = f(2,1)$.
If $n \geq 3$,

$$\begin{aligned}
f(n,2) &= \frac{n!}{2} \sum_{k=1}^{n-1} \frac{1}{k(n-k)} \\
&\geq \frac{n!}{2}\left(\frac{1}{1(n-1)} + \frac{1}{(n-1)1}\right) \quad \text{(first \& last terms of the summation)} \\
&= \frac{n!}{2} \frac{2}{n-1} \\
&= \frac{n!}{n-1} \\
&\geq \frac{n!}{n} \\
&= (n-1)! \\
&= f(n,1)
\end{aligned}$$

Proof 2: (by induction on n)
Base case $(n = 2) : \ f(2,2) = 1 \geq 1 = f(2,1)$
Inductive step: Assume for some m, $f(m,2) \geq f(m,1)$.
Then

$$\begin{aligned}
f(m+1,2) &= f(m,1) + mf(m,2) \quad \text{(our recurrence)} \\
&\geq mf(m,2) \\
&\geq mf(m,1) \quad \text{(by the inductive hypothesis)} \\
&= f(m+1,1)
\end{aligned}$$

(d) (Extra Credit) Note that $x(x+1)(x+2)(x+3) = 6x + 11x^2 + 6x^3 + x^4$
Conjecture and prove a relationship between $f(n,k)$ and polynomials of this type.
Proposition: The number $f(n,k)$ is the coefficient of x^k in the polynomial $x(x+1)(x+2)\ldots(x+n-1)$.
Proof: Define the generating function

$$G_n(x) = \sum_k f(n,k)x^k$$

To find this generating function, we multiply both sides of out recurrence by x^k and sum over all values of k to get

$$\sum_k f(n,k)x^k = \sum_k f(n-1,k-1)x^k + \sum_k (n-1)f(n-1,k)x^k$$

(Note that if we define $f(n,k) = 0$ if $k < 0$ then we may sum over all k). On the RHS, we get $G_n(x)$.

On the LHS, we get

$$\begin{aligned}\sum_k f(n-1,k-1)x^k + \sum_k (n-1)f(n-1,k)x^k &= x\sum_k f(n-1,k-1)x^{k-1} + (n-1)\sum_k f(n-1,k \\ &= xG_{n-1}(x) + (n-1)G_{n-1}(x) \\ &= (x+n-1)G_{n-1}(x)\end{aligned}$$

Thus,

$$G_n(x) = (x+n-1)G_{n-1}(x)$$

and if we note $G_1(x) = x$ (since $f(1,1) = 1, f(1,k) = 0$ if $k \neq 1$) then by induction,

$$G_n(x) = x(x+1)(x+2)\dots(x+n-1)$$

which proves the proposition.

Note how in this 2-variable recurrence we multiplied by x^k and summed over k. This allowed us to pull out the $(n-1)$. When we had a similar recurrence for the Stirling numbers, we multiplied by x^n and summed over all n, since in that case there was a k that we wished to pull out of a summation.