Discrete Mathematics 21-228
Midterm Exam

You may refer to any class materials (notes, textbook, references listed on the course website, homeworks, and exams). This exam is to be done alone. Do not consult with anyone except me on any of these problems. Send me email at any time if you have any questions. I will check frequently. Show your work, and justify your results. The exam is due at 2:50 in DH 2105. Good luck!

1. An integer is called *square-free* if it is not divisible by any perfect square other than 1. How many positive integers between 1 and 200 are sqaure-free? (No credit will be given for simply listing these).
2. Let a_n denote the number of ways to tile a 3 x $(2n)$ grid of squares using 2 x 1 tiles.
   (a) Derive a recurrence relation for a_n.
   (b) Solve for a_n explicitly.
   (c) Use your recurrence relation to find the generating function for a_n.
3. Let $f(n,k) =$ the number of permutations of $[n]$ with exactly k cycles.
   (a) Derive a formula for each of the following quantities:
      i. $f(n,1)$
      ii. $f(n,n)$
      iii. $f(n,n-1)$
      iv. $f(n,2)$
   (b) Derive a recurrence relation for $f(n,k)$ and use this relation to display all values of $f(n,k)$ in a triangle for $n \leq 6$.
   (c) Prove that for all $n \geq 2, f(n,2) \geq f(n,1)$.
   (d) (Extra Credit) Note that $x(x+1)(x+2)(x+3) = 6x+11x^2+6x^3+x^4$ Conjecture and prove a relationship between $f(n,k)$ and polynomials of this type.