Discrete Mathematics 21-228
Assignment #9
Due Wednesday, June 23

1. Let G be a graph with n vertices. If M is a maximum matching of G containing m edges, prove that G has an independent set of size $n-2m$.
If M is a maximum matching of G containing m edges, then there are $2m$ M-saturated vertices. Thus there are $n-2m$ M-unsaturated vertices, and these vertices form an independent set, since if there were an edge e between any pair of them, then $M \cup \{e\}$ would be a larger matching than M.

2. A graph G is called *d-regular* if every vertex has degree d. Let $G=(V,E)$ be a d-regular bipartite graph with bipartition $V = A \cup B$.

   (a) Prove that $|A| = |B|$.

   (b) Use the Marriage theorem to prove that for $d \geq 1$ then G has a perfect matching.

   (c) Prove that there exists a partition of E into sets $M_1, \dots, M_d$ such that M_i is a perfect matching for $i = 1, \dots, d$.

   (a) Since every vertex in G has degree d, and all edges in G have exactly one endpoint in each of A and B, there are $|A|d$ edges incident to A, and $|B|d$ edges incident to B, $\Rightarrow |A|d = |B|d \Rightarrow |A| = |B|$.

   (b) We already know that $|A| = |B|$. Thus it remains to show that for any $X \in A$, $|N(X)| \geq |X|$. Pick some $X \in A$ with $N(X) = Y$. Then there are at least $|X|d$ edges incident to Y, since all edges of X are incident to Y. Thus since the degree of every vertex of Y is d, and Y is an independent set $\Rightarrow |Y| \geq |X|$.

   (c) Proof by induction on d:
   If $d = 1$, the statement follows from part (b).
   Assume true for $d-1$. Then consider a d-regular bipartite graph G. From part (b), G has a perfect matching M_d. If we delete the edges of M_d from G to get G', then G' is a $(d-1)$-regular bipartite graph, whose edges can be partitioned into sets $M_1, \dots, M_{d-1}$ such that M_i is a perfect matching for $i = 1, \dots, d-1$, by the inductive hypothesis. Thus the edges of G can be partitoned into sets $M_1, \dots, M_d$ such that M_i is a perfect matching for $i = 1, \dots, d$.

3. A graph G has 7 connected components and no odd cycles. How many different ways are there to properly color the vertices black and white?
Since G has no odd cycles, it is bipartite. We saw in class that each component has exactly two colorings, corresponding to whether an initial vertex is chosen to be colored black or white. Thus since we may make this choice independently for each of the seven components, there are 2^7 possible colorings.

4. (a) If G is the Peterson Graph, use results proven in class to show that $3 \leq \chi(G) \leq 4$.

   (b) Let G_n be the graph arising from K_n (the complete graph on n vertices) by omitting the edges of a Hamilton cycle. Determine the chromatic number of G_n.

   (a) The Peterson graph G contains an odd cycle, so $\chi(G) \geq 3$ However the maximum degree in G is 3 (in fact, G is 3-regular), so by Brooks' Theorem, $\chi(G) \leq 4$. Thus, $3 \leq \chi(G) \leq 4$. (In fact, $\chi(G) = 3$.)

   (b) Assume n is even. Number the vertices of G_n from 1 to n, and assume that the deleted Hamilton cycle is $(1, 2, \ldots, n, 1)$. First note that no more than 2 vertices may have the same color, since any 3 vertices share at least one edge. Thus at least $n/2$ colors must be used. Also note that if the vertices are grouped as $(1, 2), (3, 4), (5, 6), \ldots, (n - 1, n)$ and each group is assigned a different color, then this is a proper coloring using $n/2$ colors. Thus $\chi(G_n) = n/2$ if n is even. By the same logic, $\chi(G_n) = 1 + n/2$ if n is odd.

5. Show that a tree has at most one perfect matching.
Consider two perfect matchings M and M' of the tree T. Consider $M \Delta M'$. Since M and M' are perfect matchings, $M \Delta M'$ cannot contain any paths, since this would imply that there were unmatched vertices in one of the matchings. Further, since T is a tree, $M \Delta M'$ cannot contain any cycles. Thus $M \Delta M'$ contains no edges $\Rightarrow M = M'$.

6. Let $G = (V, E)$ be a bipartite graph with bipartition $V = A \cup B$, such that $|A| = |B| = m$. Prove that if the degree of every vertex is larger than $m/2$ then G has a perfect matching.
Again we use the Marriage theorem. Supose $\exists X \in A$ s.t. $|N(X)| < |X|$. $|N(X)| \geq m/2$, since every vertex in X has degree at least $m/2$, and thus $|X| > m/2$. Consider a vertex $v \in B - N(X)$ (there must be one since if $N(X) = B$ then $|N(X)| \geq |X|$). Note that v is not adjacent to any vertex in X, and thus it can only be adjacent to vertices in $A - X$. But $|A - X| < m/2$, since $|X| > m/2$, $\Rightarrow deg(v) < m/2$. This contradiction implies that Hall's condition is met, and thus we may apply the Marriage Theorem to prove the existence of a perfect matching.