Discrete Mathematics 21-228
Assignment #8 Solutions

1. (a) Each edge contributes 1 to the degree of each of its endpoints, thus it contributes 2 to the total sum of all degrees.

   (b) The sum of the degrees of all vertices is even, from part (a). If we omit the even terms from this sum, we get that the sum of all the odd degrees is even. Thus there must be an even number of them, since the sum of an odd number of odd numbers is odd.

2. $K_{m,n}$ has a Hamilton cycle if and only if $m = n$. If the bipartition of the vertices is $A \cup B$, then the Hamilton cycle must alternately visit vertices in each part. A list of vertices drawn alternately from A and B can only include all of the vertices if $|A| = |B|$. Conversely, if $|A| = |B|$, then any complete list of vertices drawn alternately from A and B will determine a Hamilton cycle.

3. Assume G contains no cycles. Then G will have k connected, acyclic components $G_1, \ldots, G_k$ with $k \geq 1$. Since each component is connected and acyclic, it is a tree, with $|V(G_i)| = |E(G_i)| + 1$ for each i. Summing over all i, we get

$$n = \sum_{i=1}^{k} |V(G_i)| = \sum_{i=1}^{k} |E(G_i)| + 1 = |E(G)| + k$$

This implies that $|E(G)| < n$, a contradiction. Thus G must contain a cycle.

4.
$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad M(G) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Thus,

$$AA^T = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \qquad MM^T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \qquad M^TM = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

We note that $AA^T(i,j)$ is the number of neighbors vertices i and j have in common. In particular, $AA^T(i,i) = deg(i)$. $MM^T(i,j)$ corresponds to the number of common edges incident to vertices i and j. Again, $MM^T(i,i) = deg(i)$. Finally, $M^TM(i,j)$ corresponds to the number of endpoints edges i and j have in common. Note the 2's on the diagonal. Also note that your matrices may look different if you ordered the vertices or edges differently. However you should have drawn the same conclusions.

5. (a) If G is a path with four vertices, then it is connected, and so is its complement.

   (b) If G is connected we are done. If G has $k > 1$ connected components $G_1, \ldots, G_k$, then $\overline{G}$ is connected since

   - In $\overline{G}$, every vertex of G_i is adjacent to every vertex of G_j if $i \neq j$
   - If a component G_i has more than one vertex, then there is a path between any two of these vertices, since they are both adjacent to every vertex not in G_i.

   Thus there is a path between every pair of vertices in $\overline{G}$, so it is connected.