Discrete Mathematics 21-228
Assignment #7 Solutions

1. Let the random variable X denote the number if fixed points in a permutation. Let

$$X_i = \begin{cases} 1 & \text{if } i \text{ is a fixed point} \\ 0 & \text{otherwise} \end{cases}$$

Note

$$X = \sum_{i=1}^{n} X_i$$

There are $(n-1)!$ permutations which have i as a fixed point, so

$$\begin{aligned} \mathbf{E}(X_i) &= \frac{(n-1)!}{n!} \\ &= \frac{1}{n} \end{aligned}$$

for each i. Thus, by linearity of expectation,

$$\begin{aligned} \mathbf{E}(X) &= \sum_{i=1}^{n} \mathbf{E}(X_i) \\ &= n \cdot \frac{1}{n} \\ &= 1 \end{aligned}$$

2. (a) If $Pr(\text{H}) = p$, then $Pr(\text{Alice wins after } i \text{ flips}) = \binom{i-1}{24} p^{25}(1-p)^{i-25}$, since the last coin must come up heads, and there are 24 heads and $(i-25)$ tails among the other flips. Similarly, $Pr(\text{Bob wins after } i \text{ flips}) = \binom{i-1}{24} p^{i-25}(1-p)^{25}$. Thus,

$$\begin{aligned} \mathbf{E}(X) &= \sum_{i=25}^{49} i \cdot Pr(\text{the game ends after } i \text{ coin flips}) \\ &= \sum_{i=25}^{49} i \left(\binom{i-1}{24} p^{25}(1-p)^{i-25} + \binom{i-1}{24} p^{i-25}(1-p)^{25} \right) \\ &= \sum_{i=25}^{49} i \binom{i-1}{24} \left(p^{25}(1-p)^{i-25} + p^{i-25}(1-p)^{25} \right) \end{aligned}$$

If $p = \frac{1}{2}$, this reduces to

$$\mathbf{E}(X) = 2 \sum_{i=25}^{49} i \binom{i-1}{24} \left(\frac{1}{2}\right)^i$$

(b) Similarly,

$$\begin{aligned}
Var(X) &= \mathbf{E}(X^2) - (\mathbf{E}(X))^2 \\
&= \sum_{i=25}^{49} i^2 \binom{i-1}{24} \left(p^{25}(1-p)^{i-25} + p^{i-25}(1-p)^{25}\right) \\
&- \left(\sum_{i=25}^{49} i \binom{i-1}{24} \left(p^{25}(1-p)^{i-25} + p^{i-25}(1-p)^{25}\right) \right)^2
\end{aligned}$$

If $p = \frac{1}{2}$, this reduces to

$$2\sum_{i=25}^{49} i^2 \binom{i-1}{24} \left(\frac{1}{2}\right)^i - \left(2\sum_{i=25}^{49} i \binom{i-1}{24} \left(\frac{1}{2}\right)^i\right)^2$$

(c) At this point in the game, for Bob to win there must be 5 straight tails or 5 tails and one head, with the head coming in the first five tosses. Thus,

$$Pr(\text{Bob wins}) = (1-p)^5 + 5p(1-p)^5$$

If $p = \frac{1}{2}$, this reduces to

$$\begin{aligned}
Pr(\text{Bob wins}) &= \frac{1}{32} + \frac{5}{64} \\
&= \frac{7}{64}
\end{aligned}$$

Thus, with a fair coin, Bob should receive $\frac{7}{64}$ of the prize, Alice $\frac{57}{64}$ of it.

3. Denote by the r.v. X the number of people who buy eggs in a day.

(a) Since X takes only values in N, Markov's Inequality states that $Pr(X \geq t) \leq \frac{\mu}{t}$. If we set $t = 11,000$, we get

$$Pr(X \geq 11,000) \leq \frac{10,000}{11,000} = \frac{10}{11}$$

(b) Note $\sigma = \sqrt{1000} \Rightarrow 1000 = \sqrt{1000}\sigma$. Chebycheff's Inequality states that

$$Pr(|X - \mu| \geq t\sigma) \leq \frac{1}{t^2}$$

Thus, setting $t = \sqrt{1000}$, we get

$$Pr(|X - 10,000| \geq 1000) \leq \frac{1}{1000}$$

Thus

$$Pr(9000 \leq X \leq 11,000) \geq \frac{999}{1000}$$

4. (a) Recall that the *characteristic vector* of a subset $A \subseteq [n]$ is the vector $(x_1, x_2, \ldots, x_n)$, where

$$x_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{otherwise} \end{cases}$$

If A is a randomly chosen subset of $[n]$, then $Pr(x_i = 1) = \frac{1}{2}$ for each i. Now, the only way there can be an element in $A \cap B$, but not in C is if there is some i such that the i^{th} entry in the characteristic vectors for A and B is 1, but the i^{th} entry in the characteristic vector for C is 0. The probability of this happening for any particular i is $\frac{1}{8}$, and so the probability that it doesn't happen is $\frac{7}{8}$. Thus the probability that it doesn't happen for any i is $\left(\frac{7}{8}\right)^n$.

(b) Let $m \leq (\frac{8}{7})^{n/3}$. We'll use the probabilistic method to prove the statement. Choose $A_1, A_2, \ldots, A_m$ at random. Let BAD denote the set of bad events (i.e. the events where we choose subsets such that there exist distinct i, j, k s.t. $A_i \cap A_j \subseteq A_k$.) Let $BAD(i, j, k)$ denote the events where $A_i \cap A_j \subseteq A_k$, and note that

$$BAD = \bigcup_{\text{distinct } i,j,k} BAD(i, j, k)$$

Now, given distinct, i, j, k, from part (a) we know that $Pr(BAD(i, j, k)) = \left(\frac{7}{8}\right)^n$. Thus, Boole's Inequality implies

$$\begin{aligned}
Pr(BAD) &= Pr\left(\bigcup_{\text{distinct } i,j,k} BAD(i, j, k)\right) \\
&\leq \sum_{\text{distinct } i,j,k} Pr(BAD(i, j, k)) \\
&= \binom{m}{3}\left(\frac{7}{8}\right)^n \\
&= \frac{(m)_3}{3!(m-3)!}\left(\frac{7}{8}\right)^n \\
&\leq \frac{\left((\frac{8}{7})^{n/3}\right)^3}{3!(m-3)!}\left(\frac{7}{8}\right)^n \\
&= \frac{1}{3!(m-3)!}\left(\frac{8}{7}\right)^n\left(\frac{7}{8}\right)^n \\
&= \frac{1}{3!(m-3)!} \\
&< 1
\end{aligned}$$

We conclude that since the probability of a bad event is < 1, then there is a nonzero probability of a good event, i.e. there is at least one good event. Thus there are some subsets $A_1, A_2, \dots, A_m$ s.t. $A_i \cap A_j \nsubseteq A_k$ for all distinct i, j, k.