Discrete Mathematics 21-228
Assignment #7
Due Tuesday, June 15

1. If an element of $[n]$ is mapped to itself by a permutation, it is called a *fixed point* of the permutation. If a permutation on $[n]$ is chosen uniformly at random, what is the expected number of fixed points of this permutation?

2. Alice and Bob play the following game: A coin is flipped. If it comes up heads, Alice wins, if tails, Bob wins. The first one to accumulate 25 points wins a prize.

   (a) Let the random variable X equal the number of times the coin is flipped before the game ends. Give a formula for $\mathbf{E}(X)$.

   (b) Give a formula for $\mathrm{Var}(X)$.

   (c) During the game, Alice has accumulated 23 points, Bob 20. Unfortunately they have lost their coin. What is a fair way to divide the prize given the current state of the game?

3. Suppose the number of people who buy eggs in a day is a random variable with an expected value of 10,000, and a variance of 1000.

   (a) Use Markov's inequality to obtain an upper bound on the probability that more than 11,000 people will buy eggs on a particular day.

   (b) Use Chebycheff's inequality to obtain a lower bound on the probability that between 9,000 and 11,000 people will buy eggs on a particular day.

4. (a) Let A, B, and C be sets chosen uniformly and independently at random from the collection of all subsets of $\{1, \ldots, n\}$. Set up a probability space that describes this experiment, and prove that

   $$Pr(A \cap B \subseteq C) = (\frac{7}{8})^n$$

   Hint: Think of A, B and C as strings of 0's and 1's.

   (b) Let $m < (\frac{8}{7})^{n/3}$. Use Boole's inequality to show that there exist sets $A_1, \ldots, A_m \subseteq \{1, \ldots, n\}$ such that for all distinct i, j, k we have

   $$A_i \cap A_j \nsubseteq A_k$$

   Hint: Choose $A_1, \ldots, A_k$ at random.