Discrete Mathematics 21-228
Assignment #6 Solutions

1. You are dealt a poker hand: a set of five cards drawn at random from a standard deck. Describe the probability space for this experiment. Determine the probabilities for the following events:

   (a) You are dealt a flush (all cards are of the same suit).

   (b) You are dealt four of a kind (4 cards of the same value, with one other card, e.g $\{4, 4, 4, 4, 9\}$)

   (c) You are dealt three of a kind (3 cards of the same value, with 2 other cards of different values which are also different from each other, e.g $\{4, 4, 4, Q, 9\}$)

   (d) You are dealt a full house (3 cards of the same value, with 2 other cards which have the same value as each other, e.g $\{4,4,4,Q,Q\}$)

   The probability space consists of

   $$\Omega = \{\text{five-card hands}\}$$

   with uniform distribution, i.e.

   $$Pr(\omega) = \frac{1}{|\Omega|} = \frac{1}{\binom{52}{5}}$$

   for each $\omega \in \Omega$.
   Since the distribution is uniform, we only need to count the number of each type of hand to determine its probability.

   (a) There are four suits, each of 13 different cards. Thus there are $\binom{13}{5}$ flushes of each suit, and $4\binom{13}{5}$ total flushes. Thus,

   $$Pr(\text{flush}) = \frac{4\binom{13}{5}}{\binom{52}{5}}$$

   .

   (b) There are 13 different values for the four-of-a-kind. For each of these, there are 48 different possibilities for the fifth card in the hand. Thus there are $13 \cdot 48$ different four-of-a-kind hands, and

   $$Pr(\text{four of a kind}) = \frac{13 \cdot 48}{\binom{52}{5}}$$

   (c) There are 13 values for the 3-of-a-kind, and $\binom{4}{3}$ ways to pick the three cards. For the fourth card, there are 48 choices, and 44 choices for the fifth card. However we will pick every pair twice in this scheme, so we must divide by 2. Thus there are a total of $13\binom{4}{3}48 \cdot 44 \cdot \frac{1}{2}$ 3-of-a-kind hands, and

$$Pr(\text{3-of-a-kind}) = \frac{13\binom{4}{3}48 \cdot 44 \cdot \frac{1}{2}}{\binom{52}{5}}$$

(d) There are $13\binom{4}{3}$ ways to choose the 3-of-a-kind, and $12 \cdot \binom{4}{2}$ ways to then choose the pair. Thus there are $13\binom{4}{3} \cdot 12\binom{4}{2}$ full houses, and

$$Pr(\text{full house}) = \frac{13\binom{4}{3} \cdot 12\binom{4}{2}}{\binom{52}{5}}$$

2. Alice and Bob play a game in which they alternately toss a pair of fair dice. The one who is first to roll a total of 7 wins the game. What is the probability that the person who goes first wins the game?
The person who goes first will win if there are an even number of non-sixes rolled before a six is rolled. The sum of these probabilities is

$$\sum_{k=0}^{\infty}(\frac{5}{6})^{2k}\frac{1}{6} = \frac{\frac{1}{6}}{1-(\frac{5}{6})^2} = \frac{6}{11}$$

3. You are given a (fair) 6-sided die. To win a prize, you can either try to:

   (a) roll at least one 6 in 6 rolls.

   (b) roll at least 2 6's in 12 rolls.

   (c) roll at least 3 6's in 18 rolls.

   Which option should you choose?

   (a)

$$\begin{aligned} Pr(\text{success}) &= 1 - Pr(\text{failure}) \\ &= 1 - (\frac{5}{6})^6 \\ &\approx .6651 \end{aligned}$$

   (b)

$$\begin{aligned} Pr(\text{success}) &= 1 - Pr(\text{failure}) \\ &= 1 - (\frac{5}{6})^{12} - \binom{12}{1}\frac{1}{6}(\frac{5}{6})^{11} \\ &\approx .6187 \end{aligned}$$

(c)

$$\begin{aligned} Pr(\text{success}) &= 1 - Pr(\text{failure}) \\ &= 1 - (\frac{5}{6})^{18} - \binom{18}{1}\frac{1}{6}(\frac{5}{6})^{17} - \binom{18}{2}(\frac{1}{6})^2(\frac{5}{6})^{16} \\ &\approx .5973 \end{aligned}$$

Thus we should choose option (a).

4. Two identical-looking coins are placed in front of you. You are told that one of the coins is fair, and the other comes up heads 3/4 of the time. You pick up one of the coins and flip it 5 times, getting 3 tails and 2 heads. What is the probability you picked up the fair coin? Hint: Bayes' Theorem.
   Let "fair" = the event that you pick the fair coin, "unfair" = you pick the unfair coin. Note $Pr(\text{fair}) = .5 = Pr(\text{unfair})$.
   Further, note that

$$Pr(3T, 2H|\text{fair}) = \binom{5}{2}(.5)^5 = .3125$$

   and

$$Pr(3T, 2H|\text{unfair}) = \binom{5}{2}(.25)^3(.75)^2 \approx .0879$$

   By Bayes' Theorem,

$$\begin{aligned} Pr(\text{fair}|3T, 2H) &= \frac{Pr(3T, 2H|\text{fair})Pr(\text{fair})}{Pr(3T, 2H|\text{fair})Pr(\text{fair}) + Pr(3T, 2H|\text{unfair})Pr(\text{unfair})} \\ &\approx \frac{(.3125)\cdot(.5)}{(.3125)\cdot(.5) + (.0879)\cdot(.5)} \\ &\approx .78 \end{aligned}$$

5. Let X and Y be Poisson random variables with parameters λ_1 and λ_2, respectively. What is $Pr(X + Y = n)$? (You may assume the events $\{X = i\}$ and $\{Y = j\}$ are independent.) Hint: Binomial Theorem.

$$\begin{aligned} Pr(X + Y = n) &= \sum_{k=0}^{n} Pr(X = k, Y = n - k) \\ &= \sum_{k=0}^{n} Pr(X = k)Pr(Y = n - k) \quad \text{since } \{X = i\} \text{ and } \{Y = j\} \text{ are independent} \\ &= \sum_{k=0}^{n} e^{-\lambda_1}\frac{\lambda_1^k}{k!}e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!} \\ &= e^{-(\lambda_1+\lambda_2)}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\lambda_1^k\lambda_2^{n-k} \\ &= e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!} \end{aligned}$$

Note that this is the same as $Pr(Po_{(\lambda_1+\lambda_2)} = n)$.