Discrete Mathematics 21-228
Assignment #5 Solutions

1. To prove
$$b(n) = \sum_k \binom{n-1}{k} b(k)$$
recall that $b(n)$ counts the number of ways to partition a set of $n$ elements. To show that the RHS counts the same thing, consider for each partition of $[n]$, how many elements are *not* in the same partition class as $n$. If this number is $k$, there are $\binom{n-1}{k}$ ways to choose these $k$ elements, and $b(k)$ ways to partition them once they are chosen. Thus there are $\binom{n-1}{k}b(k)$ partitions of $[n]$ with $k$ elements not in the same partition class as $n$. Summing over all posible values of $k$, we get the total number of possible partitions of $[n]$, which is the RHS.

2. (a) To get a recurrence for $f(n)$, consider all subsets which include $n$. These subsets may not include $(n-1)$, but may include any subset of $[n-2]$ which does not contain consecutive elements. Thus there are $f(n-2)$ such subsets. Now consider the subsets which do not include $n$. These subsets include any subset of $[n-1]$ which does not contain consecutive elements, and so there are $f(n-1)$ such subsets. We conclude:
$$f(n) = f(n-1) + f(n-2)$$
This is the Fibonacci recurrence, and if we note that $f(0) = 1, f(1) = 2$, we see that $f(n)$ is the $(n+2)^{nd}$ Fibonacci number, $F_{n+2}$.

   (b) If $n$ is an element of such a $k$-subset, then the rest of the subset is a $(k-1)$-subset of $[n-2]$ containing no two consecutive elements. There are $f(n-2, k-1)$ of these. If $n$ is not an element of such a $k$-subset, then the subset is a $k$-subset of $[n-1]$ containing no two consecutive elements. There are $f(n-1, k)$ of these. We conclude:
$$f(n,k) = f(n-1,k) + f(n-2,k-1)$$
Define the generating function
$$G_k(x) = \sum_{n \geq 0} f(n,k)x^n$$
Note
$$\begin{aligned} G_1(x) &= \sum_{n \geq 0} f(n,1)x^n \\ &= \sum_{n \geq 0} nx^n \\ &= \frac{x}{(1-x)^2} \end{aligned}$$

If $k > 1$, then if we multiply both sides of our recurrence by $x^n$ and sum over all valid values of $n$, we get

$$\sum_{n\geq 2} f(n,k)x^n = \sum_{n\geq 2} f(n-1,k)x^n + \sum_{n\geq 2} f(n-2,k-1)x^n$$

The LHS $= G_k(x) - f(1,k)x - f(0,k) = G_k(x)$. On the RHS,

$$\begin{aligned}\sum_{n\geq 2} f(n-1,k)x^n + \sum_{n\geq 2} f(n-2,k-1)x^n &= x\sum_{n\geq 2} f(n-1,k)x^{n-1} + x^2\sum_{n\geq 2} f(n-2,k-1)x^{n-2} \\ &= x\sum_{n\geq 1} f(n,k)x^n + x^2\sum_{n\geq 0} f(n,k-1)x^n \\ &= x(G_k(x) - f(0,k)) + x^2 G_{k-1}(x) \\ &= xG_k(x) + x^2 G_{k-1}(x)\end{aligned}$$

So we get

$$\begin{aligned}G_k(x) &= xG_k(x) + x^2 G_{k-1}(x) \\ \Rightarrow G_k(x) &= \frac{x^2}{1-x} G_{k-1}(x) \\ \Rightarrow G_k(x) &= \frac{x^2}{1-x} G_{k-1}(x) \\ \Rightarrow G_k(x) &= (\frac{x^2}{1-x})^{k-1} G_1(x) \\ &= (\frac{x^2}{1-x})^{k-1} \frac{x}{(1-x)^2} \\ &= \frac{x^{2k-1}}{(1-x)^{k+1}}\end{aligned}$$

Now,

$$\begin{aligned}\frac{1}{(1-x)^{k+1}} &= \sum_n \binom{n+k}{n} x^n \text{ (by binomial expansion)} \\ \Rightarrow \frac{x^{2k-1}}{(1-x)^{k+1}} &= x^{2k-1}\sum_n \binom{n+k}{n} x^n \\ &= \sum_n \binom{n+k}{n} x^{n+2k-1}\end{aligned}$$

Thus, $f(n+2k-1,k) = \binom{n+k}{n}$. We conclude:

$$\begin{aligned}f(n+2k-1) &= \sum_k f(n+2k-1,k) \\ &= \sum_k \binom{n+k}{n} \\ &= \sum_k \binom{n+k}{k}\end{aligned}$$

Re-indexing, we get:

$$\begin{aligned} f(n) &= \sum_k \binom{n-(2k-1)+k}{k} \\ &= \sum_k \binom{n+1-k}{k} \end{aligned}$$

(c) Combining the results in parts (a) and (b), we get:

$$F_{n+2} = \sum_k \binom{n+1-k}{k}$$

Re-indexing, we conclude

$$F_{n+1} = \sum_k \binom{n-k}{k}$$

The binomial coefficients on the RHS of the equation are found on the "shallow diagonals" of Pascal's Triangle.

3. Let

$$f(x) = \sum_n a_n x^n$$

Then

$$\begin{aligned} f(x) &= (1+x+x^2+\cdots)(1+x^5+x^{10}+x^{15}+\cdots)(1+x^{10}+x^{20}+\cdots) \\ &= \frac{1}{1-x}\frac{1}{1-x^5}\frac{1}{1-x^{10}} \end{aligned}$$

4. Let

$$\begin{aligned} A &= \{\text{numbers between 1 and 3000}\} \\ A_1 &= \{\text{numbers between 1 and 3000 divisible by 5}\} \\ A_2 &= \{\text{numbers between 1 and 3000 divisible by 7}\} \\ A_3 &= \{\text{numbers between 1 and 3000 divisible by 11}\} \end{aligned}$$

Then

$$\begin{aligned} |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| &= |A| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - |A_1 \cap A_2 \cap A_3| \\ &= 3000 - (\lfloor\frac{3000}{5}\rfloor + \lfloor\frac{3000}{7}\rfloor + \lfloor\frac{3000}{11}\rfloor) + (\lfloor\frac{3000}{35}\rfloor + \lfloor\frac{3000}{55}\rfloor + \lfloor\frac{3000}{77}\rfloor) - \lfloor\frac{3000}{385}\rfloor \\ &= 300 - (600+428+272) + (85+54+38) - 7 \\ &= 1870 \end{aligned}$$