Discrete Mathematics 21-228
Assignment #5
Due Wednesday, May 2

1. In class we used generating functions to prove that the Bell numbers $b(n)$ satisfy the following recurrence:

$$b(n) = \sum_k \binom{n-1}{k} b(k)$$

for $n \geq 1, b(0) = 1$. Give a direct combinatorial proof of this result.

2. (a) For an integer $n$, let $f(n) =$ the number of subsets of $[n]$ that contain no two consecutive elements. Find the recurrence that is satisfied by these numbers, and describe the numbers themselves.

   (b) For given integers $n, k$, let $f(n, k)$ be the number of $k$-subsets of $[n]$ that contain no two consecutive elements. Find the recurrence that is satisfied by these numbers. Then find a suitable generating function to find the numbers themselves. Show the numerical values of $f(n, k)$ in a triangular arrangement for $n \leq 6$.

   (c) Compare the results of part (a) and (b) to deduce an identity. Draw apicture of the elements of Pascal's triangle that are involved in this identity.

3. Let $a_n =$ the number of ways to make change for $n$ dollars using 1-, 5-, and 10-dollar bills. Find the generating function for $a_n$.

4. How many numbers between 1 and 3000 are not divisible by 5, 7, or 11?