Discrete Mathematics 21-228
Assignment #4 Solutions

1. Let a_n denote the number of such strings of length n. Note there are 2^n strings of length n. For a string of length $n+1$, we fix the first bit:

   - If the first bit is 1, then there are a_n possibilities, since the remaining bits must form a string of length n with an even number of zeros.
   - If the first bit is 0, then there are $2^n - a_n$ possibilities, since the remaining bits must form a string of length n with an odd number of zeros (i.e. an n-bit string *without* an even number of zeros).

   Thus $a_{n+1} = a_n + 2^n - a_n = 2^n$.

   Note that exactly half the strings have an even number of zeros. By flipping the last bit, we can make a bijection between strings with an even number of zeros and strings with an odd number of zeros .

2. Consider a string of length $n+1$ with no two adjacent digits identical. The first n digits must be a string of length n with no two adjacent digits identical. The last digit must be different from the n^{th} digit, so there are two possibilities for this last digit. Thus if a_n denotes the number of n-digit strings with no two adjacent digits identical, we get the recurrence $a_{n+1} = 2a_n$. Since $a_1 = 3$, $a_n = 3 \cdot 2^{n-1}$

3. This recurrence has characteristic polynomial

   $$q^3 - 2q^2 - q + 2 = (q-1)(q+1)(q-2)$$

   Thus the characteristic roots are $1, -1$, and 2, and the general solution is

   $$a_n = b_1 + b_2(-1)^n + b_3 2^n$$

   The initial conditions give us the system of equations

   $$\begin{aligned} b_1 + b_2 + b_3 &= 6 \\ b_1 - b_2 + 2b_3 &= 6 \\ b_1 + b_2 + 4b_3 &= 18 \end{aligned}$$

   whose solutions are $b_1 = 1, b_2 = 1, b_3 = 4$
   Thus $a_n = 1 + (-1)^n + 4 \cdot 2^n$

4. We get the recurrence

   $$H_n = 2H_{n-1} + 1$$

   since to move a tower with n disks, we must first move a tower with $n-1$ disks (H_{n-1} moves) then move the largest disk (1 move), then move the tower with $n-1$ disks again (H_{n-1} moves).
   To solve this recurrence, we note that it is a nonhomogeneous, linear

recurrence relation. Thus we must guess a particular solution. Looking at the first few values of H_n, 1, 3, 7, 15, we guess that the solution is $H_n = 2^n - 1$. Since our particular solution is identical with the actual solution, there is no need to add a solution of the associated homogeneous linear recurrence relation.

5. The number of solutions will be equal to the coefficient of x^{16} in the expansion of

$$(1+x+x^2+x^3+x^4)(1+x+x^2+x^3+x^4+x^5+x^6)$$
$$(1+x+x^2+x^3+x^4+x^5)(1+x+x^2+x^3+x^4+x^5+x^6)$$

There are 55 solutions.